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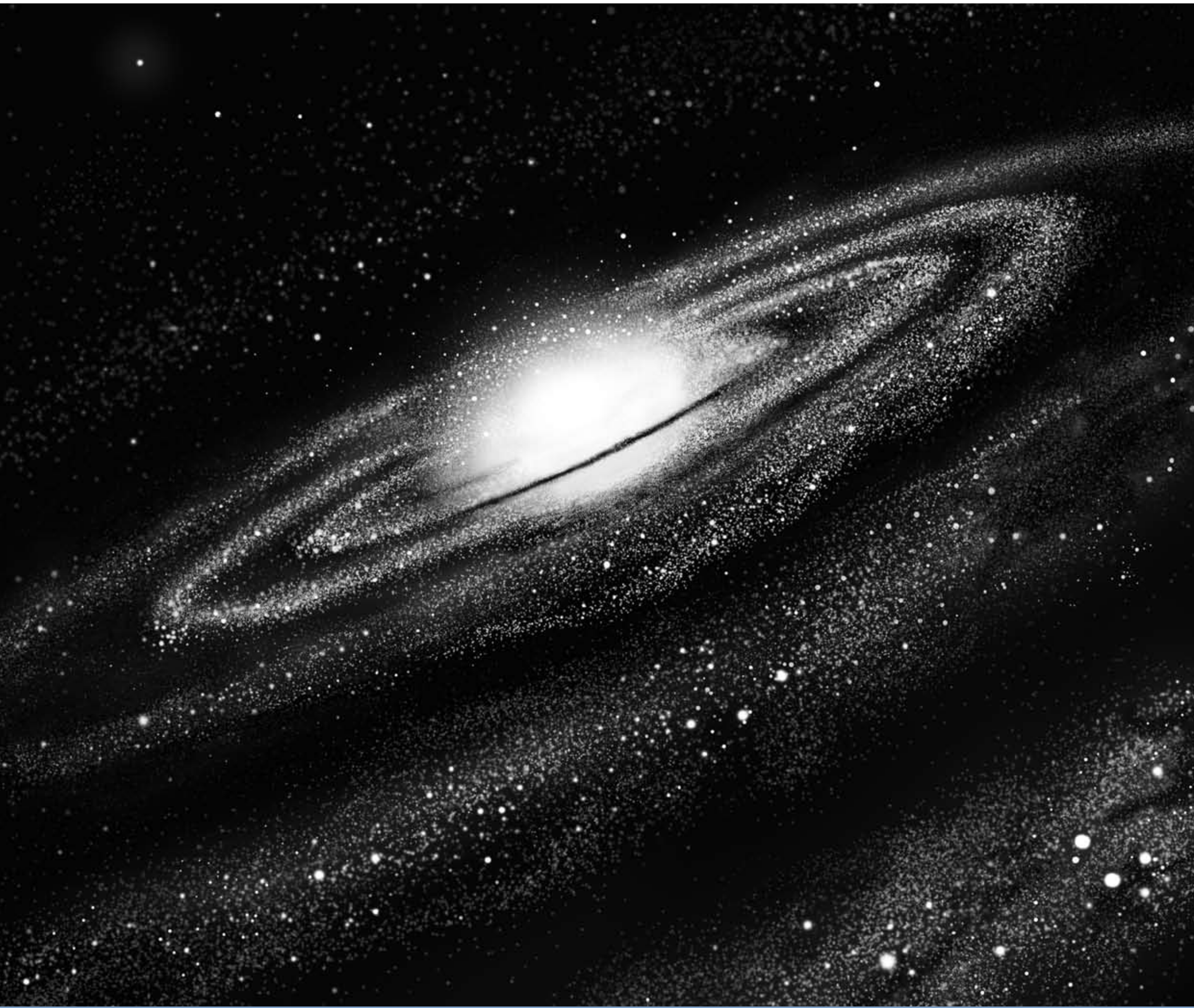
TIMSS 2007
Swedish Pupils' Mathematical Knowledge



Skolverket

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Beställningsadress:
Fritzes kundservice
106 47 Stockholm
Telefon: 08-690 95 76
Telefax: 08-690 95 50
E-post: skolverket@fritzes.se
www.skolverket.se

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Foreword

TIMSS *(Trends in International Mathematics and Science Study)* is an assessment of students' knowledge in mathematics and science in grades 4 and 8. In addition to the students' subject knowledge, information on national regulations and goals, on organization and instruction, and on students' attitudes towards the subjects are collected by questionnaires. The study facilitates comparisons between countries and gives information on trends in students' knowledge in the subject contents measured.

In Sweden TIMSS 2007 is carried out by the Swedish National Agency for Education in co-operation with didactics at the University of Gothenburg. The Swedish National Agency for Education has written a first national report from TIMSS 2007, report 323. The agency has the ambition to use TIMSS-data in different in depth analysis to contribute to school development.

This report is written as part of the TIMSS 2007 project. In the report students' answers to TIMSS-items in mathematics are analyzed. The focus is on the content domains where Swedish students do less well. These domains are *Number* and *Geometric shapes and Measures* in grade 4, *Algebra* and *Geometry* in grade 8. The analysis aim to show how well students understand key mathematical concepts and can apply calculation-procedures in these domains. To strengthen the results from the analysis, similar items from the national assessment in grade 5 are analyzed. In the final part of the report the author sums up the measures he considers has to be taken to improve students knowledge in mathematics.

The analysis is carried out and the report is written by Per-Olof Bentley, associate professor and PhD at the University of Gothenburg, as responsible for the mathematics education part in the Swedish TIMSS 2007-project. The author is responsible for the contents and the statements expressed.

Stockholm, November 2008

Per Thullberg	*Camilla Thinsz Fjellström*
Director General	Director of Education

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Introduction

1. Introduction

When the TIMSS-project of 2007 was being planned, the ambition was, besides the descriptive report, to analyse pupils' ways of solving the different test items. The result arrived at would make it possible to take high precision measures for enhancing pupils' achievements. General training of mathematics teachers would probably not be an effective way for enhancing pupils' mathematical development. Instead, more specific training with its point of departure in the pupils' difficulties would be preferable.

Within the TIMSS-project, a pilot study was performed in the spring of 2006. During the assessment of the pupils' solutions to the different test items we noticed that certain mistakes were more frequent than others. The solving frequency of several items was well below 50 %. Our observations were confirmed by the results of the main study performed in the spring of 2007. Taking these experiences into account, it was decided that an analysis of the pupils' solutions of those test items that were to be published in 2008 would be carried out for the two grades 4 and 8. The preliminary experiences of the pupils' mistakes were strengthened by this analysis, whose result will be accounted for in the present study. It was shown that certain of the mistakes were not only repetitive but also frequent. This was not, however, a sufficient basis for drawing any conclusions about the origins of the mistakes.

A possibility to make a detailed description of the origins of the pupils' different mistakes emerged when I and Magnus Olsson, a head master in the community of Lilla Edet, planned an in-service training project in mathematics for the teachers in two of its schools. It deserves to be pointed out that this contact was established within the TIMSS-project. Besides improved mathematical knowledge, the ambition was to enhance the social climate in the schools. Some scientific results show that successful learning implies that pupils comprehend their classroom situation as more positive, which in turn leads to improved social climate (Arzt & Armour-Thomas, 1999). The point of departure of the in-service training project was a scientific examination of the obstacles to the pupils' mathematical development. Altogether about 300 pupils of grades 1, 2, 3, 4 and 7 were in-depth interviewed. It was not long before it became clear that the main obstacles were to be found in the pupils' understanding of the number concept within the arithmetic domain. Swedish pupils' arithmetic achievements in grade 4 were below the average level of the EU/OECD-countries. Decisive parts of the results of the Lilla Edet-study will be accounted for in the research review of the present study in order to make it possible to deduce the pupils' different ways of reasoning in the exposed mistakes in the TIMSS-items. Together with the results from previous research, the causal factors behind the pupils' mistakes were possible to determine with high precision. It was shown that only half of the pupils in grade 7 had acquired calculation procedures that were optimal for working memory and thus for continuous learning. The results from neuroscience mathematics education research have shown the decisive role of working memory and of its functional parts in pupils' development of arithmetic knowledge.

In order to secure the external validity of the TIMSS-results, about 500 pupils' solutions to the national assessment test of grade 5, 2007 have been analysed.

For the other three domains in which Swedish pupils' achievements were below the EU/OECD-average, the analysis takes its starting point in the results of the international research in mathematics education.

Concerning the two domains of geometry for grade 4 and 8, several international studies cast light on pupils' understanding of key geometric concepts. A number of these studies will be accounted for in the research review. Also knowledge within the domains of arithmetic and algebra constitutes a basis for the development of pupils' knowledge of geometry. Therefore the analysis of these two domains can also cast light on the analysis of geometry.

Within algebra, grade 8, equivalence, letter notifications and the variable concept play a decisive role. These are the basis for pupils' understanding of a number of important concepts like equations, expressions, functions, formulas and graphs. In combination with arithmetic knowledge, the variable concept constitutes a fundamental part within the algebraic domain of knowledge. Therefore, special attention will be paid to pupils' understanding of the equality sign and the variable concept. In the research review, a number of classical studies within this domain will consequently be reported.

Previously in-depth analysis of pupils' understanding has not been made to such a large extent. Hence, in order to improve the pupils' mathematical learning situation only a few measures have been possible to take. These measures, however, have not been particularly precise. Accordingly, in-service training programmes for teachers have had a general focus on mathematics. Instead, a more specified set of measures would have been preferable for a more effective improvement of the pupils' learning situation in mathematics.

Theoretical Framework

2. Theoretical Framework

The starting point for the analysis of the pupils' solutions concerning the test items in TIMSS 2007 and in the national assessment test for grade 5 is an extended Phenomenographic theoretical framework within the Post-positivist paradigm (for a more specific account see Bentley (2008a; 2008b) and Marton and Booth (2000)).

2.1 The Learning of Concepts and Procedures

Within Phenomenography both concepts and procedures are seen as phenomena. Previously learnt concepts play an important role when new concepts are being experienced. If a new concept will be possible to experience, it needs to be discerned from previously learnt concepts. This occurs when critical conceptual attributes are discerned and comprehended, a process that takes place when the characteristics of the attributes are being varied. In this way, variation gets a central role in the process of experiencing. Some conceptual attributes are critical in the process of distinguishing one concept from another. Different individuals, however, do not necessarily experience one and the same attribute as critical (Bentley, 2008a; Marton & Booth, 2000).

Repeated exposure to the attributes of a concept plays a crucial role in the learning of it. Two partly different processes "theory revision" and "redescription" operate at different frequency levels of exposure to the concepts.

At low-frequent exposure, the learning takes place by *"theory revision"*. First, a conceptual prototype is created, which can be a relatively rough conception. At repeated exposure the conception is refined successively and over time approaches a conception that is in harmony with the individual's flow of motor and sensory data. The description of this process is to a high degree similar to the Vygotskian theory of learning every-day concepts (Bentley, 2008a).

The process of *"redescription"* on the other hand operates at high-frequent exposure to the concept in question. The conception is formed in the associative parts of the brain and is checked at the individual's flow of motor and sensory data before it is stored by redescription in long-term memory (Bentley, 2008a).

A concept is seen to be understood, when sufficient knowledge has been acquired about the conceptual attributes and about the other involved concepts. Also knowledge about the relationship between the attributes and those other concepts within the wholeness of the new concept is necessary (Bentley, 2008a, p. 11).

In school mathematics, simplifications of the concepts of the mathematical discipline are used. Such simplifications, which are made in order to facilitate the pupils' learning, are called conceptual models and often have limited application ranges. A lot of these models that are used in Swedish schools have been paid attention to in international research (Bentley, 2008a).

Procedures, which are also seen as phenomena, have an analogous structure compared to concepts. When a concept is built up by attributes and other known concepts, a procedure is built up by previously known procedures that constitute the different steps within it. These are often strictly sequenced and are linked within the wholeness of the procedure. A procedure could be applied both correctly and incorrectly. Especially the incorrect application is interesting, since it can reveal an individual's understanding. From a scientific perspective, it is more rewarding to study how a procedure is applied in a context rather than to determine whether the applications are correct or incorrect. If a procedure is correctly applied in an adequate context, it is said that the individual masters the procedure. The models of procedures of the discipline of mathematics that are used in school mathematics are sometimes referred to as procedural models (Bentley, 2008b).

In a problem text, a situation is always described by words. The solving of the problem requires that a mathematical model is created, which also describes the situation but in numbers. This process is termed encoding. The mathematical model comprises at least one operation. In order to arrive at such a model, it is necessary for the pupils to be familiar with the four elementary operations, addition, subtraction, multiplication and division. It is also needed that the pupils have the ability to evaluate the character of the problem situation (Bentley,2008a).

Theoretically, a problem situation is to be seen as a concept with its conceptual attributes, which describe the character of the situation. A problem situation could be exemplified by a comparison in which two numbers are to be compared. The very comparison then characterises the situation. To a certain type of problem situations a specific operation is linked. So, the encoding comprises the identification of the problem situation described by the text and the determination of the operation linked to it (Bentley, 2008a).

2.2 The Analysis of Data

Within Phenomenography data is traditionally seen as transcribed interviews. It is assumed that the individuals' spoken language reflects their ways of understanding different phenomena. Also individuals' behaviour has been taken as a point of departure for Phenomenographic analysis (Lindahl, 1996). By comparative analyses of data, categories that represent ways of understanding are first created on group level. The reason for this is that the stability is higher on group level than on the individual level. This is due to the fact that an individual can expose not only fragments of ways of understanding but also more or less complete ways. From a fragment it could be hard to create a complete category. When the categories, which epistemologically are to be regarded as scientific knowledge, are described, the analysis could return to the individual level. From the data, it is then identified which ways of understanding each single individual has exposed. An individual can both hold a number of ways of understanding one and the same concept and apply different calculation procedures for one and the same operation. This makes the variation two-dimensional, one that qualitatively describes the character of the ways of understanding and another that describes the variation of the number of ways each individual has exposed. The latter can therefore be submitted to statistical analysis. Each one of the individuals in a sample holds a number of ways of understanding or ways of applying procedures. In the TIMSS-project, however, one pupil could expose only one solution to each test item. This can be described statistically by means of frequencies and relative frequencies, which imply that conclusions about the character of the population can be drawn from the results achieved in the sample (Bentley, 2008a).

In certain cases the ways of understanding could seem incompatible. It is important to notice that an individual could hold a number of ways of both understanding and applications than exposed in the interviews. The interview and the character of the questions determine whether a certain conception or procedure is exposed. Thus the learning process and its results are seen from the learner's perspective within a Phenomenographic frame work (Bentley, 2008a).

When pupils' solutions of the test items in the TIMSS-project and in the national assessment test are analysed, the solutions are regarded as documented behaviour of the applications of the calculation procedures and are founded on different ways of understanding. The precision in the documentation varies, which affects the interpretation accordingly. Even if a solution is well documented, it can be hard to link it causally to a specific way of understanding or application of a procedure. If a particular way of understanding is previously known, the causal analysis will be facilitated. Further, if pupils have been interviewed about their ways of understanding and their ways of applying procedures in relation to a specific type of problems, a casual relation could more easily be identified. These preconditions make the interpretation process decisively easier. However, an interpretation could be deceptive if a solution fortuitously coincides with what only seems to be an application of a certain procedure. If on the other hand large groups of pupils solve a particular item in the same way, this is hardly only dependent on chance. Instead, it is probably due to a special application of a certain procedure or an application of a procedure not adapted to the particular context.

Taking the abovementioned into consideration, an adequate procedure for analysing pupils' solutions of test items in mathematics could be to interview a number of pupils about their solving strategies, about their ways of understanding the concepts involved and about their applications of procedures. When the descriptive categories have been created, it is possible to draw conclusions about non-interviewed pupils' solutions. The present study is therefore founded on the assumption that pupils' solutions reflect their ways of understanding concepts and their ways of applying procedures.

In those mathematical domains where no interviews have been performed as a preparation for the analysis of the pupils' solutions, international and national research is considered to have the corresponding role.

2.3 Reliability and Validity

Reliability is a measure of the accuracy of the data collection process (Swedner, 1978). High reliability presupposes a relaxed interview situation, non-guiding questions and non-confirming responses from the interviewer. If pupils' solutions of test items are studied, the way the test is performed, can influence reliability.

Validity in Phenomenographic studies primarily comprises internal and external validity. Internal validity consists of both content and construct validity.

Content validity catches how well un-interpreted data describe individuals' ways of understanding concepts and individuals' applications of procedures. Therefore, questions and test items must have a character that allows multifaceted exposure of ways of understanding concepts and applications of procedures. Otherwise ways of understanding and applications can remain unexposed. Since the applications of certain procedures are contextually dependent, the test items must contain a certain variation. The content validity is therefore a criterion of authenticity (Bentley, 2008a).

Construct validity, on the other hand, focuses the result of the interpretation of data and is therefore a measure of the extent to which the categories reflect the pupils' understanding of concepts and applications of procedures given the data at hand (Bentley, 2008a).

External validity concerns the concept of generalization. In an extended Phenomenographic theoretical frame work, the concept of generalization has a somewhat different meaning than in quantitative studies. The descriptive categories are considered to represent the variation that is possible to find in the population. Since the present study is carried out in an extended Phenomenographic frame work, a statistical analysis of data is also carried out. In this part of the present study, the concept of generalization has partly a different meaning and concerns the possibility to extrapolate conclusions from the sample to the population. Whether the sample is representative or not becomes a crucial issue (Bentley, 2008a).

The present study comprises three samples from the spring semester of 2007. The first two samples consist of pupils, who participated in the TIMSS-test for grade 4 and 8. The third sample was those pupils whose solutions to the national assessment test for grade 5 had been collected nationally. It is important to cast light on the issue of extrapolating these three results from the samples to the respective population, which comprises all pupils of grade 4, 5 and 8.

The selection of the pupils who participated in TIMSS was made on school level. A fortuitous selection of 155 schools for grade 4 and 159 for grade 8 was made. Since the selection was made on school level, individual pupils were not fortuitously selected.

Not all solutions of the test items have been analysed but mainly those that were to be released. For each grade the released test items consist of 6 tests out of totally 14 and correspond to 43 % of all items. In grade 4, about 450 pupils have participated in each test according to a rotated design of the TIMSS-project. For grade 8, the corresponding figure was 550 pupils. Altogether, nearly 33 000 pupil solutions of grade 4 and nearly 40 000 of grade 8 have been analyzed. Taking this into account, only clear tendencies in the performance of the tests have been analysed.

Pupils' solutions in the national assessment test for grade 5 are about 500, which is a relatively small proportion of the population of about 100 000. The basis of 500 pupils' solutions is too small for independent conclusions. However, if the same type of mistakes or profiles of knowledge could be found in more than one study, the possibility to generalize will be more secure.

The Limiting Role of Working Memory

3. The Limiting Role of Working Memory

The most common functional model of working memory comprises three parts, the central executive function, the phonological loop and the visual-spatial sketchpad. The model is developed by Baddeley and his colleagues (Baddeley, 1986; 1996; Baddeley & Hitch, 1974; Logie, 1995). The central executive performs operations and fetches data from long-term memory. It also directs attention and coordinates the activities of the phonological loop and of the visual-spatial sketchpad. The phonological loop that decodes speech-sounds into words, sentences and meanings works constantly without interruptions. In arithmetic, it stores the involved numbers and their partial results. Visual-spatial information, however, is stored in the visual-spatial sketchpad, where arithmetic problems and their results can be visually represented. Thus, each one of these functions has a specific role when arithmetic calculations are being performed (Adams & Hitch, 1998).

Number facts, which are not facts but rather a skill, are fetched by the executive function from long-term memory. Consequently, there is no extra load on the two other functions and therefore it is possible to store more facts in them. Rapidity is also a decisive factor, since the content of these two functions decline within a short while (DeStefano & LeFevre, 2004).

Concerning single-digit arithmetic problems, the central executive is always engaged, while the engagement of the phonological loop seems to be determined by the character of the calculation procedure. When for example the Jumping strategy or the Compensating strategy is applied, the cognitive load increases. If on the other hand number facts is used, the load will become less or none. So far, the visual-spatial sketchpad seems not to have been examined thoroughly enough and its role is therefore uncertain. In multi-digit operations, the engagement of the central executive is large and increases in proportion to the number of carries that are involved (DeStefano & LeFevre, 2004).

If pupils have not developed number facts, the cognitive load on their working memory will be high when performing calculations. The consequence of this is that attention can not be directed to anything but the very calculation per se. If the teaching concerns a geometric content and a calculation is involved, pupils that have not developed number facts will have to direct a large part or their whole attention to the calculation itself and little or none to the geometric content (Bentley, 2008b).

Pupils' Conceptual Understanding – Some Decisive Research Results

4. Pupils' Conceptual Understanding – Some Decisive Research Results

In the first section, pupils' development of their understanding of the number concept will be described. After that the variable concept, another key concept, will be paid attention to. Different ways of understanding the variable concept will be accounted for in the second section. Thirdly, not only will pupils' understanding of a number of geometrical concepts be reported but also the teaching of these concepts.

4.1 Pupils' Development of Their Understanding of the Number Concept

First, the contextual meanings of the number concept will be treated. Then children's different developmental steps towards full understanding of the number concept will be accounted for. This description is based on a research review by Fuson (1992). How different types of situations in text problems can be modelled by means of the four elementary operations, addition, subtraction, multiplication and division will then be analyzed. Lastly, the different ways of understanding the concept of proportionality will be described.

4.1.1 The Contextual Meaning of the Number Concept

The number concept has seven different contextual meanings. The *cardinal* meaning is referred to when the number word is used in order to give the number of objects in a set. The cardinal meaning also represents the numerosity of the number concept. In contrast, the *ordinal* meaning implies that each number word is a name for each corresponding object in an ordered sequence of objects. Besides, the ordinal meaning simultaneously represents a description of the relative position of an object in the sequence. The third mathematical meaning is a *measure* of a continuous quantity in a measurement context. Hereby, the measure is linked by the unit to the quantity. Fourthly, the number word is used in a *sequential* meaning without any objects present as in the rattling off the alphabet. If, on the other hand, there are objects present, we will get a context in which the numbers have a *counting* meaning and where each number has a one-to-one correspondence to the objects. In a *digit* context, numbers can be coded by a digit code or by a language code. Lastly, in a *category* context, numbers are used as telephone numbers, various kinds of addresses, bus numbers, etc. (Fuson, 1992).

4.1.2 The Different Developmental Steps

First, children can rattle off the number sequence without being able to distinguish each number word. This is similar to rattling off meaningless syllables. Gradually, the different number words are distinguished. But it occurs that the order is changed or that a number word is omitted. Children's order of the number sequence could be more or less private and remain so for a long time if not paid any attention by the teacher. Children can for example leave out a certain number word of the number sequence and are able to perform calculations that for the uninformed person seem incorrect but are consistent with the child's own number sequence. If for instance the child leaves out the number eight, then three and five can make nine, since nine has the function of the left out eight. This seemingly mistake is actually an internal consistence of the child's thinking. When such mistakes appear, an adequate measure to take would be to check the child's number sequence (Fuson, 1992).

The next step in the development of the understanding of the number concept comprises the understanding of the ordinal aspect that is the understanding of the relative order of objects. Generally, children can relate a number word to an object. The relative position of an object is given by the related number word. As a rule, the implicit one-to-one principle is mastered by children (Gelman & Gallistel, 1978). Gradually, children understand that the last spoken number word is also the answer to the question: How many? In the beginning, this transition from the ordinal to the cardinal aspect can be more or less an application of a rule (the last-number-word-rule), but little by little the understanding increases. The ability of mastering simple additions by a counting-all strategy will be reached. If three is to be added to five, a child can start counting from the beginning up to three and then continue to count five until eight is reached. When the child later is able to apply the count-on strategy and thus does not need to start from the beginning, the transition from the cardinal aspect to the ordinal aspect has become bi-directional (Fuson, 1992).

Pupils' understanding of the additive part-whole-aspect is an important milestone in their mathematical development. For example, the number eight is not only composed of five and three but also of four and four, two and six and so forth. The understanding of the part-whole-aspect has shown to be of decisive importance for children in their later development of calculation strategies (Fuson, 1992).

An often neglected aspect of the number concept is the multiplicative part-whole-aspect. The number 12 is, for example, composed of four and three but also of two and six. The understanding of this aspect has shown to be particularly decisive for the teaching of the concept of proportionality.

Place value, which is part of our decimal number system, is crucial for pupils' understanding of the number concept and is especially valid for subtractions that demand trading.

The understanding of the abstract character of the numbers, the so called abstraction principle, refers to the transformation of the understanding of a number as an adjective (two cars) into a substantive (two) of a universal meaning and denotes every two-object constellations (Gellman & Gallistel, 1978). In the example two cars, "two" has the role of an adjective that is an attributive to the substantive cars. In the constellation of an elephant and an ant, "two" denotes the number of objects. Considering these aspects, exaggerated use of concrete teaching materials can hold back the development of the abstract character of the numbers (Fuson, 1992).

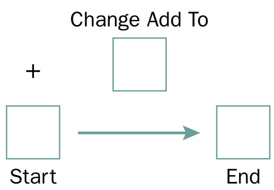
A frequent mistake among younger pupils is the reversing of the digits in a number. If a pupil for example is asked to write the number 21 and writes the number 12, then the digits of that number have been reversed. When being asked to write the number fifteen in digit code, a beginner may be confused, since the number five is not the one that is written first as is, however, indicated by the language code. The order of the digits in the number 15 is reversed compared to the language code (Bentley, 2008b). Usually, the number interval 1 to 20 is focused during the children's first school years and could be influential when it comes to reversing. This delimited number interval is probably intended to be a simplification for the beginners but does actually hinder them to get a broader picture of a larger number interval. From 20 and upwards, the digit code corresponds with the language code. Accordingly, in the number 23, the two tens come first followed by the three ones. Thus, there is a decisive difference between the numbers 10 to 20 and the numbers larger than 20. This difference in the number interval could be still more difficult for pupils with other first languages than Swedish, depending on even higher in-consistence between digit and language codes (Bentley, 2008b; Johansson, 2005).

4.1.3 The Modelling of Text Problems by Addition and Subtraction

The process of forming a mathematical model when having read a text problem is labelled encoding. According to Fuson's (1992) research review, there are three principally different problem situations to be represented by means of addition and three by subtraction. The problem situations that can be represented by addition are change, physical combination of two parts and conceptual combination of two parts.

In the change situation, the starting point is three quantities of which one is known, namely the start. A certain number is added to it and results in a new number, the end. The problem situation is schematically illustrated in figure 1.

Figure 1 The Problem Situation, Change Add To (after Fuson 1992)



An example of an unknown end in a problem situation could be: "Charles has 3 cookies. He gets another two from Stina. How many does he have then?"

In figure 2, a second problem situation represented by addition is a physical combination of two parts that together constitute a whole.

Figure 2 The Problem Situation, Physical Combination (after Fuson 1992)

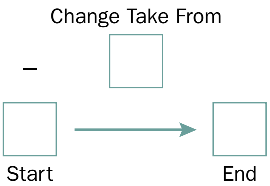
In a case of an unknown combination, a problem may be as follows: "Lena has 2 cream cakes and 3 ice-cream cakes, which she puts together on a plate. How many cakes does she have there?" The physical combination occurs, when Lena puts the cakes together on a plate.

Conceptual combination represents the third problem situation and is illustrated in figure 3. In this case, the combination is not physical but conceptual, which is shown in the example: "A soccer team comprises eight girls and seven boys. How many players are there in the team?"

Figure 3 The Problem Situation, Conceptual Combination (after Fuson 1992)

The three principally different problem situations represented by subtraction are *change take from, equalize* and *compare*. "The change-take-from situation is illustrated in figure 4. The start is a known number. By taking away a certain number, a change takes place and gives the result of a changed number, the end.

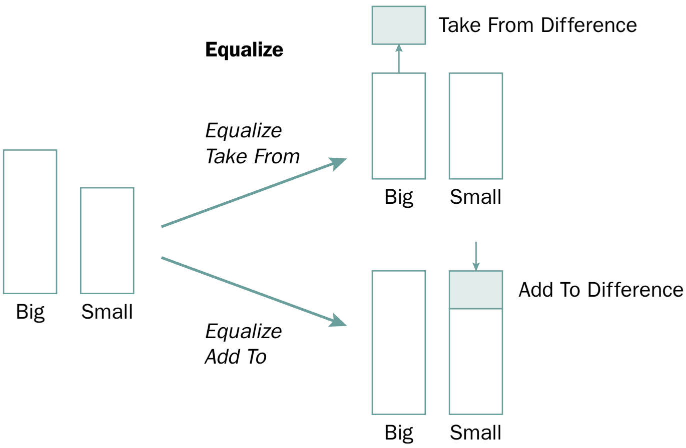
Figure 4 The Problem Situation, Change Take From (after Fuson 1992)



An example of an unknown result after the change could be: "Erik has nine marbles. He gives five to Tom. How many has he left?"

The equalize problem situation comprises two different cases, "equalize take from" and "equalize add to". Figure 5 illustrates the two situations. If two quantities differ, that is if one is bigger than the other, an equalisation can take place. If the difference is not known, a calculation can be performed in which the difference is seen as an unknown.

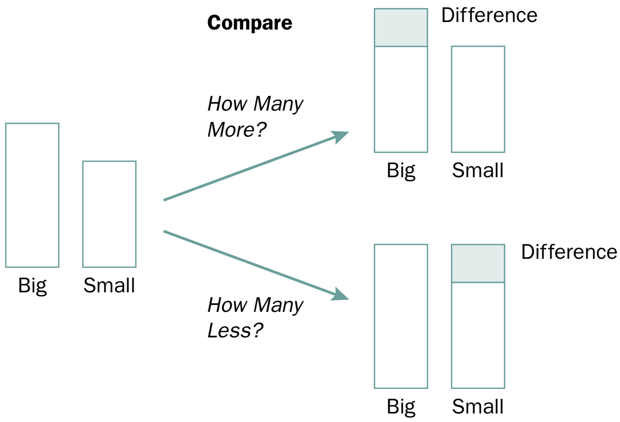
Figure 5 The Problem Situation, Equalize (after Fuson 1992)



Equalisation can be made in two different ways, first, by adapting the bigger part to the smaller or second, by adapting the smaller part to the bigger. If the result of the equalisation is unknown, the problem could be as follows: "Bill has eleven dollars and Pat has fourteen. How many dollars must Pat give away in order to have as many as Bill?"

In figure 6, a comparison situation is presented and gives answers to the two questions "How many more?" and "How many less?" In the first case, the excess part of the big quantity is seen as the difference, while in the second case, the difference is the part that the small quantity lacks. If the result of the comparison is unknown, a problem could be: "Jim has 4 balloons. His brother Nicolas has 2 balloons. How many balloons less has Nicolas than Jim?"

Figure 6 The Problem Situation, Compare (after Fuson 1992)



Fuson (1992) claims that pupils, who are trained in recognising the different problem situations, will more easily solve the corresponding problems.

4.1.4 The Modelling of Text Problems by Multiplication and Division

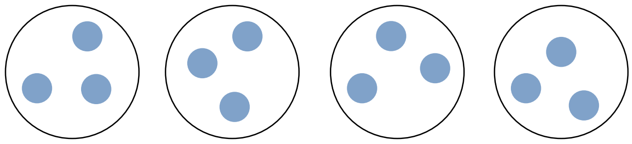
The following descriptions are based on a research review by Greer (1992). There is a similarity between the different problem situations modelled by the operations multiplication and division and those by addition and subtraction. Multiplication operations have to be regarded conceptually for sorting out the principally different problem situations. Hereby the multiplicator and the multiplicand have to be distinguished. Greer (1992) gives the following example: "3 children have 4 cookies each. How many cookies do they have altogether?" (p. 276). The two numbers, three and four, have different conceptual roles. The *multiplicand*, the number of cookies, is multiplied by the *multiplicator*, the number of children. Consequently, from a conceptual perspective there is an asymmetric relationship between the two factors in multiplication.

Due to the relationship between the multiplicand and the multiplicator, two types of division can be distinguished. The number of children can be regarded as equivalent to the number of groups of cookies. Then the total number of cookies has to be divided by the number of groups in order to get the number of cookies in each group. This is represented by *partitive division*. If, however, the total number of cookies is divided by the number of cookies in each group, it will represent a quotitive division, which is linked to the measurement principle. In order to measure the length of a distance, the number of measurement units to cover the distance has to be found out.

In figure 7, the situation of equal-groups is illustrated. Modelling it by multiplication, the problem can be formulated: "4 children have 3 apples each. How many apples do they have altogether?" Four has the role of multiplicator and three the role of multiplicand.

Partitive division, a model of the problem situation, could be derived from the following example: "There are 12 apples that 4 children shall share. How many do they get each?" This problem is modelled by a division by the multiplicator.

Figure 7 Multiplication and Division Situation ”Equal-Groups” (Greer 1992)



Quotitive division, on the other hand, is derived if the total number of apples is divided by the multiplicand: “There are 12 apples. How many children can get 3 apples each?”

Multiplicative change situations can be represented by multiplication or division depending on what is known from the text problem. The following problem can illustrate this: “A rubber band, which at the start is 3 dm, can be extended to 4 times its original length. What is its extended length?” The problem is schematically described in figure 8. In this case, four is the multiplicator and three the multiplicand.

In order to be modelled by a partitive division, the problem can be formulated accordingly: “A rubber band can be extended 4 times its original length to get its extended length 12 dm. What was its original length?” The extended length is divided by the multiplicator four. If, in contrast, the extended length 12 dm had been divided by the multiplicand 3 dm, the model would have represented a quotitive division.

Figure 8 Multiplication and Division Situation, ”Change” (Greer 1992)

A change situation can be modelled by an addition, a subtraction, a multiplication or a division depending on the formulation of the problem. The following problem can be represented by either a subtraction or a quotitive division: “The extended length of a rubber band is 12 dm. Its original length was 3 dm. Determine the size of the change!”

Comparison situations can be modelled by multiplications or divisions. A comparison situation is exemplified as follows: “Pia has 2 pairs of shoes and Siv has 3 times as many. How many pairs of shoes does Siv have?” The multiplicator is three and the multiplicand two. In order to be modelled by a partitive division, the problem must be formulated as follows: “Siv has 3 times as many pairs of shoes as Pia. Siv has 6 pairs of shoes. How many pairs of shoes does Pia have?” The total number of pairs is divided by the multiplicator three. This model represents a partitive division. In contrast, if the encoding is to result in a quotitive division, the problem is exemplified as: “Siv has 6 pairs of shoes and Pia 2 pairs. How many times more pairs of shoes does Siv have?” In this case, the number of shoes that Siv has is divided by the number Pia has, which is also the multiplicand.

Calculation of the area of a rectangle ends up in a multiplicative model as in the problem in figure 9.

Figure 9 Multiplication and Division Situation, "the Area of a Rectangle" (Greer, 1992)

The task is to calculate the area of the rectangle whose height is 2 dm and base 4 dm. In this example, there is no conceptual difference between 2 dm * 4 dm and 4 dm * 2 dm, which implies that no distinction between partitive and quotitive division is possible, or even meaningful. It is to be noticed that the calculation of the area of the rectangle is not the same as the calculation of its perimeter, which is done by measuring it. Measuring the area of a rectangle, however, is done by counting the number of area units that are needed to cover the rectangle. This is conceptually represented by a quotitive division.

Fischbein, Deri, Nello and Marino (1985) investigated pupils' mistakes concerning the encoding of text problems. The pupils had to choose suitable operation models for the different problem situations. Only single operations were allowed. The result implied that Fischbein et al. suggested the following theory:

> ***Each basic arithmetic operation remains linked to an implicit unconscious and primitive model. The identification of the model, which is needed to solve a problem comprising two numbers, was not performed directly but was mediated by the model. The model transformed its limitation on the encoding process (p. 4).***

Repeated addition can serve as such a model for multiplication as long as it only comprises integers. If it, however, were a decimal number, this primitive model would not work. Hence, it would not be possible to solve the problem correctly.

Concerning partitive and quotitive division as models of problem situations, Fischbein et al. claimed that partitive division is the original and primitive model for division and that quotitive division is acquired by the pupils much later and then due to instruction. There are several studies that cast light on this issue. A number of pupils have for instance been encouraged to formulate text problems to the operation of division. From this was shown that a waste majority of the pupils formulated text problems that resulted in partitive division and not in quotitive division (Af Ekenstam & Greger, 1983; Bell, Fischbein & Greer, 1984; Kaput, 1985; Mangan, 1986). There are also clear indications of similar behaviour from studies on a number of primary teacher students (Graeber & Tirosh, 1988). From this it would be possible to draw the conclusion that pupils acquire quotitive division relatively late during their educational training.

4.1.5 Pupils' Pre-Instructional Understanding of Addition of Fractions

Davis (1997) investigated a number of lessons with focus on teacher-pupil interaction in primary school. His documentation shows that pupils, who had not been taught how to operate with addition of fractions, spontaneously added not only the nominators but also the denominators. Instead of using the correct procedure and getting
$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, the incorrect $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$ was arrived at.

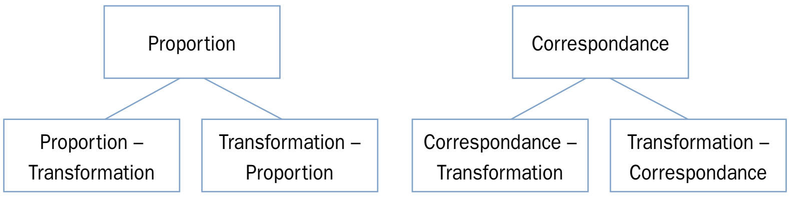
However, when the teachers started to affect their pupils' understanding of addition of fractions operation, the pupils seemed to acquire it after a sequence of lessons (Davis, 1997).

4.1.6 Pupils' Understanding of Proportionality

The description of the proportionality concept and of the ways of understanding it is based on Bentley's study (2008a). First, it is important to bear in mind that proportionality is comprised of either a multiplicative change or a multiplicative comparison. Actually, a proportionality problem is made up by two situations, one in which the multiplicative change is presented and another in which this change is applied in the second situation. Two main ways of understanding can be distinguished. The first concerns a relation between one part and the whole and the second a relation between two parts. An example of the first relation can be expressed as 2 parts of 8 parts, which means that the 2 parts are included in the 8 parts, which is the whole. The second relation, however, can be expressed as 2 parts to 6 parts, where all the parts together constitute the whole, namely 8 parts. But two quantities could also be related to each other. The part-whole relation is labelled *proportion*, while the relation between parts or quantities is labelled *correspondence*. In figure 10, teachers' ways of understanding the concept of proportionality are displayed.

Below, the two categories proportion and correspondence are more thoroughly described. The basis for the description of the category of proportion is the so called lemonade problem: "You are about to mix lemonade. On the bottle it says 2 parts of lemonade to 6 parts of water. You have 9 litres of water in a bucket. How much lemonade shall you add?"

Figure 10 Categories of Teachers' Exposed Understanding of the Concept of Proportionality



The understanding of proportionality as a proportion is manifest by the fact that the teachers discern that 6 parts and 2 parts together constitute the whole, namely 8 parts. The different proportions of lemonade and of water are expressed as 6 eighths and 2 eighths and are schematically illustrated in figure 11.

Figure 11 Conceptual Understanding of Proportion

The relation between 6 and 8, in the first situation, can be described by the multiplicative changing factor 4/3. Applying the factor in the second situation gives 4/3 * 9 = 12. Then the total amount of mixed lemonade is 12 litres and the amount of concentrated lemonade is 3 litres.

On the other hand, the understanding of the concept of proportionality represented by the category correspondence is based on a relation between two parts of the same whole. Again proceeding from the lemonade problem, the 2 parts of lemonade correspond to the 6 parts of water. Together they make up the whole of 8 parts. In figure 12, the problem is schematically illustrated.

Figure 12 Conceptual Understanding of Correspondence

First situation	2	→	6
Second situation	?	→	9

The correspondence is fully determined in the first situation:
2 * k = 6, which gives k = 3.

By this calculation, the correspondence is known in its main aspects. The amount of water to be added to a certain amount of lemonade is three times the amount of lemonade. In the second situation, the amount of water is known, namely 9 litres. Then the amount of lemonade becomes 3 litres.

4.1.7 A Frequently Used Conceptual Model for Proportionality

Hart (1981) described a conceptual model of proportionality, which was partly introduced in Great Britain. The model has also frequently been used in Sweden.

It is written: $\frac{a}{b} = \frac{c}{d}$ and is read: "*a* is related to *b* as c is related to *d*".

As any conceptual explanation to the division included in the model is usually not given, the model is often procedurally applied. The model, which can be applied to problems linked to both proportion and correspondence, does not, however, make any difference between the two. This fact can bring about difficulties for the pupils to discern what numbers the variables stand for. Instead, the formula should be understood as a description of a calculation.

It is worth noticing that proportionality problems generally have low solving frequencies.

4.2 Pupils' Understanding of Key Concepts in Algebra

The two concepts of decisive importance for mastering algebra are the concept of variable and the concept of the equality. These two concepts will be dealt with in the following sections.

4.2.1 The Concept of Variable

The meaning of the variable concept is determined by the context of which it is part and which contains its letter denotation. The most common misconceptions of variable are caught by the four categories Non-symbolic representation, Digit representation, Concrete object representation and Specific unknown number (Küchemann, 1981; Wagner, 1983; Booth, 1984; Philipp, 1992; MacGregor & Stacey, 1997; Bentley, 2008a).

Non-symbolic representation represents an understanding where the letter denotation is not seen to have any meaning and is therefore ignored. Simplifying the expression $3a + 2a$, only the coefficients are taken into consideration and it equals five.

The second way of understanding the concept of variable is as a *Digit representation.* Given $2b$, where $b = 8$, the result arrived at is not 2 times 8 but the number 28. This way of understanding is especially frequent in notations where the multiplication signs are left out.

The understanding characteristic of the *Concrete Object representation* category is as a concrete object and has its roots in a conceptual model used for teaching algebra. This model, whose application range is rather narrow, is frequently used to simplify addition of variables. In the example $3a + 2b + 4a$, the letter "*a*" stands for apples and "*b*" for bananas. The different fruits are added separately. In multiplicative expressions, however, the model is not possible to apply. The multiplication $2a * 3a$ would according to the model become two apples multiplied by three apples. This, however, lacks conceptual meaning and thus does not support the development of pupils' understanding of the concept of variable, nor does it give any operational guidance.

The abovementioned three ways of understanding the concept of variable that from a normative perspective could be possible to denote as misconceptions are more or less independent of the context. Understanding the concept as a *specific unknown number* is also contextually independent, which it, however, should not be. In equation contexts, it is correct that the variable symbolizes a specific unknown number. But it is of importance to emphasize that the understanding of the concept of variable as a specific unknown number also appears in quite different types of contexts, in which the variable has quite different meanings like in functions, graphs and equations with two unknowns.

It seems relatively unusual that the variable in the aforesaid contexts is understood as a generalized number and as simultaneously representing several numbers and an infinite quantity of numbers. A common way of expressing this is to say that the variable represents each one of all numbers. Understanding the variable likewise has shown to facilitate pupils' understanding of functions and graphs.

Another way of understanding the concept of variable is as an *abbreviated quantity.* When formulas are taught, abbreviated quantities are often introduced. For example, when the formula for the area of a rectangle is explained, it can run as follows "the area is equal to the base multiplied by the height, which could be written by means of abbreviations as $A = b * h$." The understanding implied by this introduction means that b is seen as an abbreviation of a word and not as a variable. The area formula can therefore not be dealt with within the structural algebra and it is thus impossible from the formula to divide both sides by b, since b is an abbreviation of a word.

4.2.2 The Concept of Equality

Also pupils' ways of understanding the equality sign are of decisive importance for their possibilities to solve equations in algebra. A lot of pupils comprehend the equality sign as a signal to do something, that is as an operator. This is usually referred to as the *dynamic* understanding. A lot of younger pupils have experienced that when calculating an addition like $3 + 5$, the answer is written to the right of the equality sign, 3 + 5 = 8 and is read as "it becomes". This way of coping with the equality sign is frequent in primary school. Despite comprehensive measures in order to improve the situation, the dynamic understanding unfortunately seems to be stable and still remains in the secondary classes, where it causes difficulties when equations are to be solved. A *static* understanding of the equality sign is required for being able to solve equations with unknowns on both sides of the equality sign. It is for example impossible to solve the equation, $3x + 2 = 4x - 3$, if the equality sign is comprehended as "it becomes", since the expression $3x + 2$ can never be simplified into the expression, $4x - 3$. For a certain value on x, however, it is possible for the two expressions to have the same value, $3 * 5 + 2 = 4 * 5 - 3$ (Ginsburg, 1977; Behr, Erlwanger & Nichols, 1980; Kieran, 1981; Falkner, Levi & Carpenter, 1999; Carpenter, Franke & Levi, 2003).

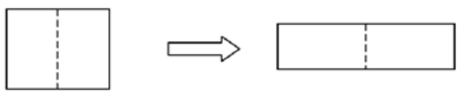
4.3 Pupils' Understanding of Geometrical Concepts

From an instructional perspective, area and perimeter are the first two geometrical concepts to be analysed. Then, the conceptions of angle together with three of its conceptual models are dealt with. Finally, the training of spatial ability with the help of computers and its effects on pupils' understanding of two-dimensional representations of three-dimensional objects will be accounted for.

4.3.1 The Concepts of Area and Perimeter

A number of Australian teachers' ability to explain the concepts of area and perimeter was investigated by Chick and Baker (2005). The nine teachers were exposed to fictitious pupils' understanding and were encouraged to describe how they would handle their mistakes. In the item, the additive character of the area concept was to be focused. First, the teachers had to make a written account of their ways of handling the pupils' ways of understanding. Later on, they were interviewed. One of the participating teachers did not understand the problem per se.

A student cuts the following shape in half to make a new shape, saying that the two shapes have the same area and perimeter:



What would you say to this student?

Most of the teachers gave an explanation, which was both procedural and conceptual in its character. Some teachers were not capable of giving any explanation at all, while one teacher gave a procedural explanation. This shows that not all the nine teachers had the competence to teach the concepts of area and perimeter satisfactorily. No general conclusions can of course be drawn due to the small sample.

Children often use the length of one side when they compare the sizes of various kinds of rectangles. Cuneo (1980) showed that children sometimes used the sum of the length and breadth as a measure of the size of a rectangle. Also Clements and Stephan (2003) found that younger children often used a linear distance as a measure of the size of a rectangle, compared to older children, who took advantage of a two-dimensional multiplicative structure

Children's difficulties to understand the area concept seem to be dependent on the fact that there are almost no activities in the teaching that focus on the conservation of the area concept. Different areas are not manipulated by being taken apart and put together in new ways. Also the premature utilisation of formulas for calculating the areas of different figures causes difficulties for too many children (Kordaki & Potari, 1997; Piaget, Inhelder & Sheminska, 1981).

A rapid transition in teaching from a conceptual study of areas to numerical calculations makes pupils develop techniques to get at the answers and does not convey any conceptual understanding (Hiebert, 1981; Douady & Perrin, 1986). Such a largely procedural approach, which is frequent in school mathematics, was criticized by Patronis and Thomaidis (2008).

In TIMSS' video study, there are recordings that illustrate those findings. Of special interest were two recordings of lessons on the understanding of the calculation of the area of the triangle carried out by one Japanese teacher and one American teacher (Stigler & Hiebert, 1999).

An initially presented problem was the starting point of the Japanese lesson during which the pupils were asked about what kinds of triangles they were familiar with. Isosceles, equilateral and right angled triangles were kinds that the pupils mentioned. Then the teacher taped up various paper triangles on the whiteboard. The pupils were encouraged to calculate the area of one paper triangle each and were given a piece of cardboard paper and a pair of scissors before starting to work independently. Having finished, they were one at a time invited to the whiteboard to present their solutions. In cooperation with the teacher, the formula for calculating the area of the triangle emerged and became clear to the pupils. After that the pupils practiced independently to calculate the areas of triangles of various forms and sizes (Stigler & Hiebert, 1999).

The American lesson, however, started with the teacher giving a short survey of the concept of perimeter. Thereafter, a calculation of the area of a rectangle was made by counting its area units. Having given the formula for calculating the area of a rectangle, the teacher showed some examples of the calculation by means of the formula as an illustration. Then the teacher took out a cut out square unit and put it in the rectangle and showed his pupils that the number of square units was easily counted. In triangles, however, the number of square units is difficult to count. From this followed the idea of constructing a rectangle which was done by combining two right-angled triangles. After that two similar cut out triangles were put together into a rectangle and the formula for the area was derived. The application of the formula was then practiced during the pupils' seat work (Stigler & Hiebert, 1999).

The most interesting feature of the Japanese lesson was that the pupils participated actively in getting at the formula for the calculation of the area of the triangle. By doing so, the establishment of the conceptual understanding of the area of the triangle took place within the pupils.

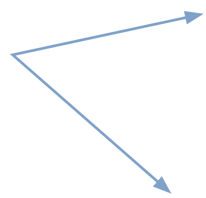
In the American class, the pupils were not equally actively engaged in the creation of the formula for the calculation of the area of the triangle. Instead, their engagement was to a rather high degree devoted to the application of the formula. This way of advancing the area of the triangle has the character of a more procedural approach.

4.3.2 The Angle Concept

The three conceptual models of the angle concept were accounted for in international research (Mitchelmore & White, 2000). The three models all comprised rays and their joint end points. Characteristic of a ray is that it starts from a point and stretches infinitely far away. Lines, however, stretch infinitely away in two opposite directions not issuing from one point. Two end points on a line constitute the limits of a segment. Therefore, a typical conceptual attribute of a segment is its length.

According to the first model, which perhaps is a little out of date, the two rays delimit two open regions (Figure 13).

Figure 13



This model has previously been used in Swedish schools and is described in ”Matematikterminologi i skolan”, the Swedish National Board of Education (1966, p. 36). There is a disadvantage with this model as it lacks operational explanation. Consequently, it does not give any guidance for the measurement of an angle.

In the second model, which does not differ decisively from the first, the conceptual image of the angle is not that of a region but of the two rays and their intersecting end points. In this way, the problem with an open infinite region is avoided. Neither does this model give any operational explanation for the measurement of an angle.

The third model, which is more dynamic, gives, however, an operational explanation for the measurement of an angle. Also in this model, the rays and their intersecting end points knock into each other. In this case, the angle is constituted by a rotation of one of the rays until they coincide. So, the rotation is the representation of the angle, whose size is determined by the rotation per se. This model takes advantage of the concept of rotation, which in turn, when defined takes advantage of the angle concept, an apparent circle definition. In spite of this fact, the model has operational advantages.

The text books in several countries utilise these conceptual models of the angle concept in school mathematics (Close, 1982; Freudenthal, 1973; Strehl, 1983; Roels, 1985; Schweiger, 1986; Krainer, 1989; Mitchelmore, 1989; Lo, Gaddis & Henderson, 1996).

The understanding and mastering of the angle concept has shown to be tricky for the pupils in several countries. Mitchelmore and White (1998) explained: ”It is clear from the research literature that school students have great difficulty in coordinating the various facets of the angle concept. For example, students do not readily incorporate turning into their angle concepts.” (p. 11). Also Foxman and Ruddock (1983) reported that only 4 % of the examined 15-year-old pupils, spontaneously mentioned rotation, when asked to describe the angle concept.

A conclusion possible to draw from research on the concept of angle is that it should have been paid more attention in comprehensive school teaching and in in-service training.

4.3.3 Spatial Visual Ability

To be able to visualise how a rotation of a three-dimensional object is depicted into two dimensions is an important component in pupils' spatial visual ability, according to McGee (1979). This ability is practiced in many countries by means of special software, which facilitates for the pupils to visualise three-dimensional objects from their two-dimensional representations. Ryu, Chong and Song (2007) found, however, that also talented pupils often had difficulties to visualise a three-dimensional object in space from its two-dimensional representation.

Ben-Chaim, Lappan and Houang (1988) studied how the training by means of special software affected about 1000 pupils' spatial visual ability. Before the training, there were significant differences with respect to age, gender and socio-economic status. Pupils in grades five to eight participated in this study. A decisively improved spatial visual ability was the outcome of the teaching. Also, the differences between boys and girls were equalised. Besides, the improvement remained unaffected after one year.

Pupils’ Mental Algorithms

5. Pupils' Mental Algorithms

First the different mental algorithms, which are frequent in teaching and in text books, will be described. Then the main features of the result of the research project in Lilla Edet will be accounted for. This project constitutes a significant link to which the analysis of the results in the present study is based.

5.1 Mental Algorithms

In the first section, a survey of significant international research within this domain with its specific terminology is given. Pupils' application of different mental algorithms or calculation procedures does not always end up with a correct answer. A study important in this respect is reviewed in section 2.

5.1.1 Terminology

An algorithm is a stepwise procedure, which we apply on an item that we wish to complete (Usiskin, 1998, p.7). This definition is frequent in the international research literature. What we in Sweden sometimes refer to as "written mental calculation" is actually a form of algorithms. So, in the following, not only will the term mental calculation procedures be used but also mental algorithms.

Within research, somewhat different terminologies for mental calculation procedures exist, but the similarities between them are striking. To harmonize these various terminologies is therefore not hard to do and this harmonization takes it starting point in a report of the research project in Lilla Edet, "Pupils' Arithmetic Knowledge and the Procedural Models in Their Teaching – A Field-Study in two Schools" (Bentley, 2008b).

The first mental algorithm is called the *Jumping strategy* according to which the jumping is done step by step in ones and in tens (Klein & Beishuizen, 1998; Thompson, 1999; Heuvel & Panhuizen, 2001).

Example: $37 + 16 = [37 \xrightarrow{3} 40;\ 40 \xrightarrow{13} 53] = 53$

The intention is to facilitate the calculation by jumping to the nearest ten. This is the basic principle for all mental algorithms. The Jumping strategy is also applied to subtractions, which first are transformed into the corresponding addition.

Example: $37 - 16 = [16 \xrightarrow{4} 20;\ 20 \xrightarrow{10} 30;\ 30 \xrightarrow{7} 37;\ 4 + 10 + 7] = 21$

Fuson et al. (1997) chose a name of their algorithm that indicates the purpose of the algorithm, the Easier number strategy. Carpenter et al. (1997) called their algorithm the Compensating strategy. Yackel (2001) in turn chose the name the Counting-Based strategy, which reflects the operating mechanism in the algorithm.

In the second mental algorithm, the *Compensation strategy*, the basic principle is to round one number up to the nearest ten and then add the next term.

Finally, there is a compensation for the rounding up number and 3 is subtracted from 56 (Carpenter, Franke & Levi, 2003).

Exempel: 37 + 16 = [37 + ***3*** = 40; 40 + 16 = 56; 56 – ***3***] = 53

The Complementary Addition or Subtraction strategy is carried out in order to reach the nearest ten (Klein & Beishuizen, 1998). This strategy is similar to the Compensation strategy and so is also the Flexible Counting strategy (Heuvel & Panhuizen, 2001).

In the Transformation strategy, the item is transformed in order to facilitate the calculation. There are to versions of the algorithm, one for addition and one for subtraction. Concerning addition, a number is added to the first term so that the nearest full ten is reached. The same number is then subtracted from the second term (Seyler, Kirk & Ashcraft, 2002).

Example addition:
37 + 16 = [37 + ***3*** + 16 – ***3*** = 40 + 13] = 53

Regarding the subtraction alternative, one and the same number is either added to the two terms or subtracted from them.

Example subtraction:
64 – 27 = [64 – ***4*** – 27 – ***4*** = 60 – ***20*** – 23 – ***20*** = 40 – 3] = 37

Both versions are usually seen as a sub group of the compensating strategy.

In the *Splitting strategy*, each number is turned into tens and ones, which are dealt with separately. The last step consists in the re-combination of the partial calculations. Two versions of the addition and subtraction algorithms exist, one with trading and one without trading (Thompson, 1999; Heuvel & Panhuizen, 2001; Klein & Beishuizen, 1998).

Example: 37 + 16 = [30 + 10 = 40; 7 + 6 = 13; 40 $\underset{\uparrow}{\mathbf{+}}$ 13] = 53

In the version of addition, the partial results are added as the arrow shows. The subtraction example below shows the subtraction of the partial results. This change of operation from addition to subtraction is due to the negative partial result of the subtraction 4 – 7 = - 3.

Example: 64 – 27 = [60 – 20 = 40; 4 – 7 = –3; 40 $\underset{\uparrow}{\mathbf{-}}$ 3] = 37

Yackel (2001) termed these versions the Collection-based solution, Fuson et al. (1997) the Decompose-tens-and-ones strategy and Carpenter et al. (1997) the Combining of units separately. Accordingly, several researchers have identified the same algorithm but termed it differently.

The *Mixed strategy* is a combination of the Splitting strategy and the Compensating strategy (Fuson et al., 1997).

Example: 64 – 27 = [60 – 20 = 40; 40 – 7 = 33; 33 + 4] = 37

The mixed strategy is an elegant solution to problem of trading in subtraction, since one and the same strategy is practicable for both with and without trading.

The mental algorithms could be divided into three principally different kinds of strategies, namely the Jumping, the Compensating and the Splitting strategies (Foxman & Beishuizen, 2002). The Jumping strategy comprises not only the ordinary Jumping strategy but also the Counting-all, the Counting-on and the Counting-down strategies. Those comprised in the Compensating strategy are the ordinary Compensating, the Transformation and the Mixed strategies. Included in the Splitting strategy are the two ordinary versions, one with trading and one without.

5.1.2 The Accuracy of Mental Algorithms

Foxman and Beiszhusern (2002) re-analysed the APU-material (APU = Assessment of Performance Unit), which consists of pupils' test results from 1987. In these tests, the pupils exposed not only the degree of accuracy on a number of different algorithms but also their mental calculation strategies. The tests also comprised items measuring conceptual understanding. Depending on their results, the pupils were divided into three groups: those with well developed conceptual understanding, with moderate understanding and with less well developed understanding. The groups were then examined with respect to the utilisation of mental algorithms. It was shown that those pupils who had exposed well developed conceptual understanding of algorithms, more frequently took advantage of the compensating strategy, while those with less developed understanding more often operated by means of the splitting strategy. The average accuracy of the various algorithms showed to be 79 % for the compensating strategy, 75 % for the standard algorithm and 33 % for the splitting strategy.

Most likely, well developed understanding of the concepts number and place value will facilitate the utilisation of the compensating strategy. Whether there is a causal relation between the choice of algorithms and pupils' conceptual understanding has not been paid any attention in the study. Evidence from neuroscience shows, however, that the working memory model supports such a relationship.

5.2 Pupils' Arithmetic Skill

In this section, the research project in Lilla Edet will be described (Bentley, 2008b). Having given the introduction, the pupils' exposed calculation strategies will be described. Then the comparative analysis of their text books will be accounted for. By this analysis, the pupils' exposed calculation strategies will be compared to the corresponding strategies in their textbooks. Finally, the relation between the pupils' development of number facts and their possibilities to acquire the teaching in mathematics will be analysed.

5.2.1 Introduction

The reason for starting the comprehensive study in Lilla Edet was an in-service training in mathematics for the teachers at two of the schools. The teachers' daily teaching problems in relation to the teaching content was the point of departure of the project. Given that, the obstacles in pupils' mathematical development had to be examined and described. In-depth interviews were seen as an effective way to describe the experienced problems with high precision. Nearly 300 pupils in the grades 1, 2, 3, 4 and 7 were interviewed. The interviews were transcribed in order to facilitate the consecutive scientific analysis.

Decisive for the further analysis of pupils' solutions in TIMSS 2007 was their ways of reasoning the procedures out on a number of test items whose solutions were high-frequent. By connecting the Lilla Edet project result to the solutions on the test items in TIMSS 2007, it was possible to draw conclusions about pupils' mistakes on a more specific level.

5.2.2 Pupils' Mental Algorithms

Due to lack of space, a full and thorough report of the complete results of the Lilla Edet project is not admitted. Therefore, only the fourth grade-pupils' mental algorithms will be focused. For a more comprehensive report, Bentley's study (2008b) is referred to. The problems that were identified had emerged in grade three and still existed in grade seven. In table 1 below, an overview of the pupils' mental algorithms is presented.

None of the pupils had developed number facts (see chapter 3, p. 8). Many of the pupils utilised the Counting-on strategy with the help of their fingers. When asked to give the answer to, 51 – 48, Dave said:

> ***Do you know how I do this? I start with this [48] and count-on to that [51]. It is the same as minus. [What is the result?] Three. [Could you solve it (the subtraction) in another way?] One can count-down as well, 50, 49, 48. We have learnt that from our teacher.***

A transformation to addition and to counting on is typical features of the Jumping strategy.

A small number of pupils applied the Counting-down strategy and doing so took advantage of their fingers.

The Splitting strategy was performed correctly by two pupils, while most of the pupils consequently applied the version of the splitting strategy that is adapted for subtraction without trading to subtractions that demand trading.

Table 1 Pupils' Exposed Arithmetic Procedures, class 4, n = 16 (17)

	Number facts		Counting			Splitting Strategies			Compensation Strategies			Standard Algorithm		
Pupil [1]	**Certain**	**Uncertain**	**All**	**On**	**Down**	**Inside a.r.**	**Outside a.r.**	**Modified**	**Ordinary**	**Transform**	**Mixed**	**Ordinary**	**Modified**	**Comments**
Dan		x		x			x	x [2]		x				[2] Trading
Dave		x		x	x	x	x							
Ed		x		x	x				x		x [2]			[2] incorrect
Felicia				x						x [2]			x	[2] incorrect
Fia				x			x							
Ivar		x					x	x [2]	x	x [3]				[2] Also trading [3] incorrect
Inge		x				x	x	x [2]	x					[2] incorrect
Meg							x		x				x	
Mia				x					x [2]	x				[2] incorrect
Ofelia					x							x	x	
Qintus		x		x	x			x						
Rune							x	x [2]	x					[2] Also trading
Tim							x	x [2]						[2] incorrect
Yngve				x			x		x					
Uno		x		x	x		x		x [2]	x [2]				[2] incorrect
Urban		x			x			x [2]						[2] Also trading, incorrect
Total		**8**		**9**	**6**	**2**	**10**	**8**	**8**	**5**		**1**	**3**	

[1] Fictitious names have been used. a.r. = applicability range.

As an illustration of this incorrect application, Inge's reply to how he reasoned is given: "[How did you perform the calculation of 37 – 23?] 14. I took 30 – 20, which is 10 and then 7 – 3, which is 4. Altogether 14". Since no trading was required, the correct version was applied to the item.

Fia, on the other hand, claimed that 51 – 48: [silence] is 17. I thought that 5 – 4 is 1 and that 1 – 8 is 7." This shows that the pupil split the numbers into tens and ones and made their separate calculations. After the recombination, the result arrived at was 17. This is an illustration of the application of the incorrect version of the splitting strategy. A lot of the pupils tried to modify the Splitting strategy so that it would fit subtractions with trading. Some of the pupils (3) were successful in doing so. Urban's exposed way of reasoning will illustrate this: 21 – 17 = [20 – 10 = 10; 7 – 11 = 4] four. The calculation of the tens and ones separately is typical of the Splitting strategy. Although Urban did not discern the order of the terms in the last subtraction and wrote 7 – 11 instead of 11 – 7, he got the correct answer 4.

Half of the pupils utilised the Compensating strategy and got at the correct answer to 9+5. Uno explained his thinking: "I think that one plus nine is ten and then I have four left, so it is fourteen." To round the first number to the nearest ten and then continue the calculation is characteristic of the way of reasoning that is typical of the Compensating strategy.

Two of the pupils applied correct versions of the Transformation strategy, while some pupils mixed the versions. Dan performed a correct calculation by means of the Transformation strategy. He was asked to calculate 82 – 47. He said: It is forty ... well 45, if I take 2 – 7 then I have five left and then I take 45 – 80 and that is 35." He subtracted two from each term and reversed the subtraction. Instead of reaching 80 – 45, he got 45 – 80. But in spite of the reversed subtraction, this way of reasoning illustrates a principally correct application of the Transformation strategy.

When Ivar was asked to calculate 51 – 49, he mixed up the two versions: "What was it? Then I take one from 51 and get 50 – 50, which is zero." Ivar exposed one of the frequent mistakes that are linked to the Transformation strategy. Instead of subtracting the same number from both of the terms, he subtracted one from 51 and added one to 49. He applied the version that is intended for addition to subtraction.

Another pupil, Ed, exposed the Mixed strategy: "[How did you do when you calculated 23 – 17?]. First, I calculated 20 – 10 and then minus seven and lastly minus three [20 – 10 = 10; 10 – 7 = 3; 3 – 3] = 0". First, he calculated the tens. Then instead of calculating the ones, he took use of some modification and subtracted the ones seven and three from the tens. Had the two threes been added instead of subtracted, the calculation would have ended up correctly.

5.2.3 A Comparative Analysis of the Pupils' Mental Algorithms and Their Text Books

The pupils' exposed mental algorithms were compared to the presentations in the text books. Mainly, two different text books had been utilised in grade three and four. It is important to notice that none of the books contained any description of the contexts in which the particular strategies should be applied. Instead, only examples were given. In the text books for grade three, the Jumping strategy for subtraction was described and the subtraction was replaced by an addition. The reason for this replacement was not accounted for. So the pupils applied the strategy in an incorrect context. When calculating 17 – 7, a number of pupils got seven. They counted down to seven instead of counting down seven steps. By doing so they mixed up the number sequence with the tracking sequence. This behaviour was firmly established and the pupils calculated several arithmetical problems likewise.

The Compensating strategy was described in the text books and several pupils exposed that they had mastered the procedure of this strategy. Largely, the same was valid for the version of the Transformation strategy for subtraction.

In the text book for grade three, the two versions of the Splitting strategy were not described. Only that for subtractions without trading was explained. A lot of the pupils, however, applied it to subtraction that requires trading. Consequently, they got the incorrect results as in 51 – 49 = 18.

A conclusion possible to draw from the analysis of the text book for grade three is that its insufficient descriptions were reflected in the pupils' mistakes.

5.2.4 The Development of Number Facts

In grade seven, about half of the pupils had developed number facts, a skill that is decisive for pupils' continuous learning in mathematics. On average, each pupil exposed three mental algorithms besides the standard algorithm and the counting on and the counting down procedures. Those pupils, who had not developed number facts, at least exposed one mental algorithm that occasionally ended up in an incorrect answer. The data showed that a necessary condition for not having developed number facts was that at least one application of the mental algorithms not always ended in a correct result. Those pupils, who had developed number facts, in contrast, exposed mental algorithms that all gave correct answers. If the calculation procedures always end up in correct results, this seems to be a sufficient condition for the development of number facts (Bentley, 2008b).

Problem Formulation and Aim

6. Problem Formulation and Aim

Since the previous reported analyses of the TIMSS-results had been descriptive, a more analytic investigation was planned for the analyses of TIMSS 2007. One of the problems with such a report is that the test items have to be described in order to make the analyses understandable. In view of the fact that a number of items are released after each testing, a possibility to analyse pupils' solutions of the released test items arose. Such an analysis can give a picture of pupils' mathematical knowledge. This kind of knowledge can explain the reasons for pupils giving successful or unsuccessful solutions. In the following, the denotation "solutions" will refer to both those ending up in correct and those ending up in incorrect results.

The analysis of pupils' mathematical knowledge reflected in the results of TIMSS 2007 comprises both a specific and a general level. At the specific level, pupils' solution strategies and their frequencies are described. At a general level it is known from previous research how the understanding of concepts and the applications of procedures can affect pupils' solutions. By taking advantage of the knowledge at these two levels, conclusions can be drawn not only about the various kinds of understanding of concepts but also about the different kinds of applications of procedures reflected in the pupils' solving strategies. This picture of the pupils' mathematical knowledge thoroughly reveals the causal factors underlying the results and is therefore a solid basis for improvements.

The Lilla Edet study, in which about 300 pupils were in depth-interviewed, offered a basis needed as a complement for certain parts of the analyses of the TIMSS 2007 results.

Also the conceptual models utilised in school geometry showed to have a great impact on pupils' possibilities to solve the test items. This is in accordance with international research results. Besides, the operability of the conceptual models determines the level of difficulty of the test items.

Within algebra, the concept of variable plays a crucial role in pupils' understanding. This is described not only in international but also in national research. Also ways of understanding other key concepts in school mathematics could affect pupils' success within the domain of algebra.

The aim of the present study is therefore to relate pupils' ways of solving test items to their ways of understanding related key concepts and to their applications of key procedures in school mathematics. Hence, the present study aims at casting light both on pupils' exposed understanding of mathematical concepts and on their exposed applications of procedures. This is done by analysing pupils' solutions to the released test items of TIMSS 2007 for grade four and eight.

As the number of pupils that performed the TIMSS-test is relatively large, general conclusions are possible to draw from the tendencies that are relatively frequent. With the intention of improving the validity of the conclusions, a corresponding analysis of pupils' solutions to certain test items of the national assessment test in mathematics for grade five, 2007, was made.

6.1 Aim

Thus the aim is

- to describe pupils' solving strategies exposed both in the released test items of TIMSS 2007 for grade four and eight, and in the collected solutions of the national assessment test in mathematics for grade five.
- to try to cast light on pupils' understanding of the key concepts and also on pupils' applications of the calculation procedures as these conceptions and applications are reflected in the exposed solving strategies.

Method

7. Method

In TIMSS, there are different types of test items, which are classified according to the ways the solutions are intended to be presented. The first type of items is that of multiple-choice. In this case, the pupils are to give their results by choosing and marking one alternative out of four or five. A full operational solution is thus not required of the pupils. In the second type of items, the pupils have to give an answer to the test item. In the third type of items, it is required of the pupils not only to give an answer but also to write down their calculations.

The assessment of the multiple-choice items is relatively easy to make. Either a pupil has put an *x* in a square or not. The statistical analyses of multiple-choice data are also relatively easy to examine. The outcome is a number of frequency distributions for each item. However, fortuitous mistakes are not represented by the item distracters but mistakes with their roots in a specific conception or in a certain application of a procedure. Such kinds of mistakes have been identified in the Lilla Edet study and in previous research. They show how a certain conception and a certain application of a procedure are manifested in the pupils' solutions and answers.

It is, however, necessary to take into consideration the influence from guesses on the frequency distribution. If there are four alternatives and they are chosen fortuitously, each one of them is equally probable and 25 % of the pupils will choose each one of the alternatives. A so called Chi2-test measures the deviation from this expected frequency distribution. If a sufficiently large deviation from the expected distribution is detected, it is not due to guesses but is rather a reflection of the pupils' knowledge. Having considered this, only clear deviations have been used as a basis for the analysis of the pupils' knowledge.

The assessment of the items that only require answers has been made consistent with the international assessment instructions in TIMSS 2007. Some of the items have deviating answers, which have their roots in a specific conception or in the application of a certain procedure. Such occasional deviations have been paid attention in the assessment instructions. Concerning other items, the answers have been analysed over again in order to make it possible to describe their character and also to determine the frequencies of specific solution strategies.

Since the third type of items per definition offers a richer description of the pupils' solving strategies, a number of them have been reanalysed. Because the fact that the assessment instructions primarily focus on the given answers and not on the different kinds of solutions, the applications of the different calculation procedures and the reflections of the pupils' different conceptions have been possible to identify by means of these documented solutions.

The TIMSS-test comprises 14 different blocks of items for grade four and eight respectively. Each test comprises two of the blocks in such a way that an overlapping occurs. Six blocks in each of the two grades have been released. About 450 pupils in grade 4 and about 550 pupils in grade 8 performed each one of the six-block items. Consequently, this analysis is based on 6000 pupils' solutions.

As the analysis of the national assessment test comprises more than 500 pupils' solutions, the total population in one grade is approximately 100 000 pupils.

The D part of the national assessment test is designed in such away that one and the same subtraction appears twice, namely in two test items. First, the subtraction is tested in a text problem and soon after it is tested without any problem text. Besides, the without-problem text subtraction had some short questions about the performance of the calculation. This made it possible to compare the pupils' solving strategies in the two contexts.

In the report on the Lilla Edet study, fictitious name are used. But since the management of the schools has given the project publicity by giving parts of the result to the press, radio and TV, ordinary ethical rules have not been possible to apply.

First, the different solving strategies of the respective test items have been described. Thereafter, relevant research for each domain of the test items has been gone through in order to find evidence that shows the consequences of pupils' various conceptions and of their applications of procedures reflected in the pupils' different solving strategies. From these research results it has been possible to create a picture of the pupils' mathematical knowledge including their conceptions and misconceptions, and also their ways of applying the calculation procedures. In this way, a factor that implies a certain solution is known. Together these factors constitute sufficient conditions for the occurrence of the various solving strategies. It could, however, not be excluded that pupils come up with solution strategies due to other reasons. Bearing this in mind, it is important that only relatively frequent solution strategies are dealt with. By doing so, there will be a decrease of the probability that fortuitous mistakes will be mixed up with mistakes caused by different ways of understanding a concept or with applications of a calculation strategy in incorrect contexts. This is the reason for only analysing the underlying factors of high-frequent multiple-choice alternatives and solving strategies.

In the frequencies of the multiple-choice alternatives, left out test items are included. It is important to notice that the reasons for not trying to solve the test items are not known. This does not necessarily mean that the pupil is not capable to solve the item. Other reasons could be lack of time or motivation for participating in the testing. The re-analysis of the test items showed that a lot of pupils, who had left out a specific item, had, however, tried to solve the kinds of items when put in some other parts of the test. This does not probably imply lack of time but rather a disability to solve the item. In some cases, the test items coming after the analysed one were all blank. This has, however, been interpreted as lack of time. Therefore, the frequency of the pupils who did not try to solve the items was of interest. Especially, when a relatively large number of pupils had left out the items, it has been reported separately.

The Results – Grade-Four Pupils' Mathematical Knowledge

8. The Results – Grade-Four Pupils' Mathematical Knowledge

As is known, Swedish pupils' results on the TIMSS-items are below the EU/OECD-average in arithmetic and geometry. This is the reason for analysing and reporting the outcome of the testing in these two specific domains.

8.1 Arithmetic Problems and the Understanding of the Number Concept

The testing of arithmetic calculations mainly comprises the understanding of the number concept and of its operations involved. In the following, besides the encoding process, the operations of addition, subtraction, multiplication and division will be covered. Also simple calculations with fractions will be treated.

8.1.1. The Understanding of the Number Concept

The testing of the understanding of the number concept comprised four items. The first item (M04_01) presupposes the understanding of the place value concept.

M04_01

Which number equals 3 ones + 2 tens + 4 hundreds?

- (A) 432
- (B) 423
- (C) 324
- (D) 234

M041052

In the four multiple-choice alternatives, the digits are permutated. The most frequent alternative is also the correct one, 423, which is chosen by approximately four fifths of the pupils (79.5 %). The second most frequent alternative, 324 was chosen by approximately one seventh of the pupils (14.7 %). The alternative, 324, represents the reversed number of the correct alternative, 423. Reversing the digits in a number is known to be frequent among younger children (Johansson, 2005; Bentley, 2008b).

In the second test item (M07_03), the pupils were requested to choose one of the four numbers that in size is closest to ten.

M07_03

Which of these numbers is closest in size to 10?

Ⓐ 0.10
Ⓑ 9.99
Ⓒ 10.10
Ⓓ 10.90

M031332

The most frequent alternative was also the correct one, 9.99, which was chosen by three fourths of the pupils (75.0 %). The two distracters, 0.10 and 10.10, were almost equally frequent, 10 %. At first sight, the solving frequency of the correct alternative gives the impression that 75 % is a satisfactory figure. But as a matter of fact it is not an acceptable figure, since one fourth corresponds to a rather large number of pupils who did not solve the problem.

In the third item (M02_01), the pupils were asked to arrange the numbers according to size, from the largest to the smallest number.

M02_01

In which of the following are the numbers arranged from LARGEST to SMALLEST?

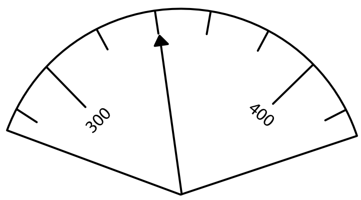
Ⓐ 36, 43, 66, 87
Ⓑ 66, 43, 36, 87
Ⓒ 87, 66, 36, 43
Ⓓ 87, 66, 43, 36

M041014

More than three fourths of the pupils (77.7 %) chose the correct alternative d). In the most frequent distracter (12.3 %) a), the numbers are given the other way around, namely from smallest to largest. A reason for having chosen distracter a) could be that the number line goes that way and is often dealt with accordingly. Therefore, distracter a) seemed natural to choose for many pupils. It is worth noticing, however, that 10.3 % of the pupils do not have the knowledge to arrange the numbers 1-100 according to size. This aspect of the number concept, referred to as numerosity, is according to Marton and Booth (2000) an often neglected part of the number concept. If the teaching has a procedural focus and the four elementary operations are dealt with more or less mechanically, numerosity will not easily be acquired.

The fourth item (M07_07) is based on a scale and a pointer. The segment between 300 and 400 is divided into five equally large parts of which each represents 20 units. Counting from 300 and upwards, the pointer indicates the second part. Four multiple-choice alternatives were presented.

M07_07



On the scale above, what number does the pointer indicate?

Ⓐ 302
Ⓑ 310
Ⓒ 320
Ⓓ 340

M031276

The most frequent alternative, 340, was also the correct one and was chosen by nearly two thirds of the pupils (57.1 %). A seemingly logical choice, 320, was the most frequent distracter and was chosen by nearly one third (30.0 %) of the pupils. The reason for the relatively high frequency of this distracter could be that the pointer indicates the second mark on the scale but also pupils' limited experience of this way of dividing intervals by marks.

8.1.2 Addition

Addition was exemplified by only one item (M01_06).

M01_06

Don, Rob, and Lynn walk home from school together. It takes them 25 minutes to walk to Lynn's house. Then it takes Don and Rob 10 minutes to get to Rob's house. From there it takes Don 5 minutes to walk home.

At what time must they leave school so that Don arrives home at 3:50 p.m.?

Answer: _______________ p.m.

M031068

In order to solve the problem, 25, 10 and 5 must be added. Then the result has to be subtracted from 15.50. Half of the pupils solved it (49.7 %), while the other half did not (50.3 %). The main difficulty seemed to be the encoding of the problem, which represents a comparison situation. According to previous research (Fuson, 1992), comparison situations are difficult for pupils to encode, unless they have thorough experience of such situations from their teaching. Also the understanding of other concepts, like time and place value, and of their corresponding procedures can have played a significant role in the solving of this problem.

8.1.3 Subtraction

Since the testing of subtraction was embedded in the problem formulations themselves, the main difficulty was not always the calculation as such but the encoding of the problem.

The first test item (M01_02) was about a missing number in the standard algorithm for subtraction.

M01_02

$$\begin{array}{r} 942 \\ -5\blacksquare7 \\ \hline 415 \end{array}$$

Mano did the subtraction problem above for homework but spilled some of his drink on it. One digit could not be read. His answer of 415 was correct. What is the missing digit?

Answer: ______________

M031106

The pupils were supposed to complete the algorithm, whose correct answer is the number 'two'. So, the difference between 942 and 527 is 415. In table 2, the categories of the pupils' ways of reasoning are displayed.

Table 2 The Categories of Pupils' Ways of Reasoning in the Calculation of 942 – 527, Grade 4, n = 461

Categories	Frequency*	Relative Frequency (%)	Typical Answer
Correct in Principle	94	20.4	527
The Splitting Strategy without Trading	250	54.2	537
Additive Miscalculation	29	6.3	557
Other Types of Incorrect Calculations	60	13.0	-
No Calculation Performed	28	6.1	-
Total	**461**	**100.0**	

* not weighted frequencies

A minority of the pupils (20.4 %) managed to solve the item correctly. A majority (54.2 %) utilised the splitting strategy without trading and got the incorrect number 'three'. In the procedural model of the splitting strategy, the hundreds, tens and ones are dealt with separately. The trading version was apparently not known by the pupils, who therefore utilized the non-trading version. This implies that the trading from the four tens to perform the calculation 2 – 7 is not paid any attention. So, one ten out of the four tens was not traded into ten ones, which would have given 12-7=5 (ones) and 3-2=1 (ten) which is also known from the item. Then the correct number 2 had easily been had.

There were also other types of miscalculations. A separate adding of the tens, for example, resulted in the number 'five'. This kind of miscalculation was, however, low-frequent.

The second test item (M01_03) was about a comparison of the number of girls in two school years.

M01_03

M031282

Last year there were 92 boys and 83 girls in Fairmont School. This year there are 210 students, and 97 are boys. How many more girls are there this year than last year? Show your work.

Answer: _______________

The encoding could lead to two consecutive subtractions, 210 – 97 = 113 and 113 – 83 = 30. So, 30 was the correct answer. Even other ways of solving the problem was exposed by the pupils. The categorisation of their ways of reasoning is displayed in table 3.

Table 3 The Categories of the Ways of Reasoning in the Pupils' Solving of the Number of Girls-Problem, Two Consecutive Subtractions, M01_03, Grade 4, n = 457

Categories	Frequency*	Relative Frequency (%)	Typical Answer
Correct Encoding and Correct Calculation	111	24.3	30
Correct Encoding but Incorrect Calculation	42	9.2	20
Incorrect Encoding but Correct Calculation	121	26.5	113
Incorrect Encoding and Incorrect Calculation	62	13.6	103
Non-Categorized	39	8.5	-
No Calculation Performed	82	17.9	-
Total	**457**	**100.0**	

* not weighted frequencies

The third test item (M02_05) dealt with subtraction of decimal numbers.

M02_05

M041250

Subtract:
5.3 – 3.8

Answer: _______________

According to the TCMA (Test of Curriculum Matching Analysis), this item is not part of the attainment targets in the syllabus. The pupils' mistakes are still of interest, because they follow the same pattern as in the items that matched the syllabus. The different ways of reasoning involved, are displayed in Table 4, below.

Approximately one third of the pupils for the most part exposed a correct conceptual understanding of place value and subtraction. A second third did not perform any calculation, while the last third exposed the version of the splitting strategy without trading, which in this case is an incorrect application.

The correct structure, in principle, comprised three different ways of solving the item. In the sample, a relatively large group mastered place value and subtraction, whilst a smaller group missed the decimal sign and got 15 instead of 1.5. Yet an even smaller group made minor miscalculation errors.

Table 4 The Categories of Pupils' Ways of Reasoning in the Calculation of 5.3 – 3.8, Grade 4, n = 461

Categories	Frequency*	Relative Frequency (%)	Typical Answer
Correct in Principle	161	34.9	1.5
The Splitting Strategy without Trading	118	25.6	2.5
Other Types of Incorrect Calculations	41	8.9	-
No Calculation Performed	141	30.6	-
Total	**461**	**100.0**	

* not weighted frequencies

One of the most frequent ways of reasoning was the incorrect application of the splitting strategy. Typical of this incorrect application is that the subtraction is done without trading. Instead, the smaller number is taken from the larger. Several pupils exposed this kind of conceptual structure. A smaller group also missed the decimal sign, while another group made minor additional miscalculations.

Other ways of reasoning that was represented were by adding the numbers instead of subtracting them and by starting the subtraction from the left in combination with incorrect trading. Some miscalculations were also to be found.

The fourth test item (M01_07) comprised, besides subtraction, a transformation of litres into millilitres.

M01_07

A bottle contains 1 liter of water. Tony pours 250 milliliters into a glass. How much water is left in the bottle?

Answer: _______________ milliliters

M031299

Nearly half of the pupils solved the problem (46.3 %), while just about one third did not (31.9 %). A little more than one fifth (21.8 %) did not try but left it out. The major obstacle in this item was probably not the subtraction per se but the transformation of the units, from litres to millilitres. The encoding of this change-take-from-situation, which is modelled by a subtraction, has not caused the pupil any particular problems.

The fifth test item (M01_08) was 'The Cat Problem'.

M01_08

Al wanted to find how much his cat weighed. He weighed himself and noted that the scale read 57 kg. He then stepped on the scale holding his cat and found that it read 62 kg.

What was the weight of the cat in kilograms?

Answer: _______________ kilograms

M031301

The item could be solved by the mathematical operation 62 – 57. In the analysis of the pupils' ways of reasoning when solving the problem, a clear distinction appeared between the process of encoding and the process of calculation. As can be seen from table 5, several pupils solved the problem correctly.

Table 5 The Categories of the Ways of Reasoning in the Pupils' Solving of 'The Cat Problem': 62 – 57', M01_08, Grade 4, n = 457

Categories	Frequency*	Relative Frequency (%)	Typical Answer
Correct Encoding and Correct Calculation	329	72.0	5 kg
Correct Encoding but Incorrect Calculation	27	5.9	15 kg
Incorrect Encoding but Correct Calculation	0	0	-
Incorrect Encoding and Incorrect Calculation	48	10.5	-
No Calculation Performed	53	11.6	-
Total	**457**	**100.0**	

* not weighted frequencies

A small group of pupils (5.9 %) arrived at the correct operation, 62 – 57, but did not succeeded in performing the calculation correctly. Most of them got the answer 15, which indicates that the splitting strategy for subtractions without trading had been utilized.

A somewhat larger group of pupils (10.5 %) did not manage to encode the problem correctly. Another group of about the same size (11.6 %) did not try and omitted the task. The problem contains a comparison situation, in which Al's weight is compared to the weight of Al and his cat. Generally, situations of this kind are hard for pupils to encode.

The sixth test item (M02_06) is about paying for things.

M02_06

M041094

Bob has 10 zeds. For lunch, he buys a bottle of fruit juice for 2.50 zeds and a sandwich for 3.85 zeds. How much money does Bob have left after he has paid for his lunch?

(A) 3.65 zeds

(B) 4.75 zeds

(C) 6.35 zeds

(D) 16.35 zeds

Three of the four multiple-choice alternatives represent different ways of encoding the problem. The first, 3.65 zeds, was the correct alternative and was reached by subtracting the prices for the juice and sandwich, 10 – 2.50 – 3.85. More than one fifth of the pupils (22.1 %) had chosen this alternative. The distracter 4.75 zeds were probably seen as a likely value but are not possible to reach by simple calculations. About one fourth of the pupils chose this alternative (27.4 %). The third alternative, 6.35 zeds, is possible to get at by adding 2.50 and 3.85. If subtracting 3.85 from 10, the difference will be 6.15, an answer that is adjacent to 6.35. Given this, it is easier to understand that a majority of the pupils had selected this alternative (42.2 %). The last alternative, 16.35 zeds, is got at when all the available numbers in the item are added, 10 + 2.5 + 3.85. A minor group of pupils (1.7 %) chose this alternative. That a large group of pupils did not choose this alternative probably depends on their thinking that Bob's sum of 10 zeds would not increase when having paid for his lunch. Instead, it would decrease.

Therefore, the core obstacle in this problem seems to be the encoding of the problem and not primarily the calculation.

The seventh test item (M07_02) was a calculation of a subtraction of two decimal numbers

M07_02

M031030

12.36 – 9.7 =

Answer: _______________

This item is not part of the attainment targets of the syllabus for mathematics grade 5. As will be seen from table 6, only four pupils of the sample managed to solve the item correctly. Nearly half of the sample utilised the without-trading Splitting strategy (48.1 %). A small group of pupils added the numbers instead of subtracting them (6.5 %). One fourth exposed ways of reasoning not possible to identify (24.5 %) and nearly one fifth did not perform any calculation but omitted the item (19.0 %).

Table 6 The Categories of Pupils' Ways of Reasoning to the Calculation of 12.36 – 9.7. Grade 4, n = 216

Categories	Frequency*	Relative Frequency (%)	Typical Answer
Correct in Principle	4	1.9	2.66
The Splitting Strategy	104	48.1	3.29
Additive Reasoning	14	6.5	21.43
Other Ways of Reasoning	53	24.5	-
No Calculation Performed	41	19.0	-
Total	**216**	**100.0**	

*not weighted frequencies

Having grasped the concept of place value and been familiar with the version of the Splitting Strategy for trading, the pupils would more or less automatically have come to the correct result.

The final test item on subtraction (M07_08) was about a baking problem.

M07_08

John is going to bake biscuits. He has to heat up the oven for 10 minutes, then bake the biscuits for 12 minutes. John wants to finish baking the biscuits at 11:00. What is the latest he should turn on the oven?

Ⓐ 10:38

Ⓑ 10:48

Ⓒ 10:50

Ⓓ 11:22

M031064

For solving the problem, 10 minutes and 12 minutes need to be added and subtracted from 11.00. The adding seemed to be a piece of cake. The subsequent subtraction, however, presupposes the ability of transforming 1 hour into 60 minutes from which 22 minutes are to be subtracted. Of the four alternatives, 10.38 is the correct answer, which also was the most frequent (56.0 %). One fourth of the pupils (25.8 %) had chosen the most frequent distracter, 10.48, which is arrived at by subtracting 12 minutes from 11.00. The other two distracters were low-frequent.

Along these lines, it seems evident that there are two major problems about the item, first the encoding of the text problem and second, the subtraction of the given times.

8.1.4 Multiplication

Multiplication was tested by six items. The encoding process had a central role in five of these items.

The first test item (M02_03) is about multiplication of whole numbers.

M02_03

M041278

Multiply:

53 × 26

Answer: ________________

Also this item was not part of the TCMA for grade 5. In table 7, the pupils' different ways of reasoning are presented. In the category 'Correct in Principle', the pupils exposed their mastering of both place value and of the multiplication operation.

Table 7 The Categories of Pupils' Ways of Reasoning in the Calculation of 53 * 26, Grade 4, n = 461

Categories	Frequency*	Relative Frequency (%)	Typical Answer
Correct in Principle	20	4.3	1 378
The Splitting Strategy	246	53.4	118
Other Types of Reasoning	79	17.1	-
No Calculation Performed	116	2.2	-
Total	**461**	**100.0**	

*not weighted frequenciesr

In a few cases, the multiplication per se was correctly done, while the adding of the partial products was done in an incorrect way, since the numbers had been placed in incorrect positions. This is illustrated in table 8, where besides the correct answer 1378, three incorrect are seen.

Table 8 Correct Multiplication of the Partial Products, the Addition of the Partial Products and the Displacement of their Positions

Displacement of the positions of the tens multiplied by 53	The ones multiplied by 53	The resulting sum	Frequency	Relative frequency (%)
106	318	424	3	
1 060	318	*1 378*	20	4,3
10 600	318	10 918	1	
106 000	318	106 318	1	

An over-generalisation of the Splitting strategy, which corresponds to a concatenated conceptual structure, was made by a majority of the pupils (53.4 %) as is evident from table 7. The numbers in the ones and tens positions were multiplied separately. Consequently, tens were only multiplied by tens and ones only by ones. By doing so, the result arrived at was 118. The reason for this is probably that the concept of place value has not been fully grasped, neither explicitly or implicitly. In contrast, the category 'Other Types of Reasoning' mostly comprises additive and subtractive ways of reasoning instead of multiplicative.

Eight children have to share a number of candies, in the next item, (M03_01).

M03_01

A group of 8 children have 74 sweets altogether. How many more sweets are needed for the children to be able to share them equally?

Answer: ________________

M031235

The categories of pupils' encoding and calculation are presented in table 9. Approximately half of the pupils (44.6 %) not only encoded the problem correctly but also calculated it correctly. One sixth of the pupils (16.4 %) encoded the problem incorrectly but produced a correct calculation. One fourth (25.7 %) left out the item.

Table 9 The Categories of Pupils' Encoding and Calculation of the 'Candy Problem', M03_01, Grade 4, n = 451

Categories	Frequency*	Relative Frequency (%)	Typical Answer
Correct Encoding and Correct Calculation	201	44.6	6 fler
Correct Encoding but Incorrect Calculation	6	1.3	
Incorrect Encoding but Correct Calculation	74	16.4	9 fler
Incorrect Encoding and Incorrect Calculation	1	0.2	-
Not Categorized	50	11.1	-
Nor Performed	116	25.7	-
Total	**451**	**100.0**	

*not weighted frequencies

From the different ways of exposed reasoning, it was obvious that the encoding of the item was the main obstacle. The problem situation is termed equal-groups. According to Fischbein et al. (1985), pupils often utilise intuitive models for encoding problems, such as the repeated addition model for multiplication. If applied to this item, this model would increase the degree of difficulty rather than decrease it.

In the third item (M03_03), Ken measures the length of the blackboard.

M03_03

Ken measured the length of a blackboard using a 30 cm ruler. The blackboard was 6 cm less than 9 times the length of the ruler. What is the length of the blackboard?

Ⓐ 264 cm

Ⓑ 270 cm

Ⓒ 276 cm

Ⓓ 279 cm

M031050

Four multiple-choice alternatives were given out of which the correct, 264 cm, was most frequently chosen (38.6 %). Some of the distracters were, however, also relatively high-frequent. One distracter represented simply 9 times the length of the ruler, 270 cm (27.8 %) and another, 276 cm, 9 times the length of the ruler plus 6 cm, instead of minus 6 cm (16.0 %). The final distracter, 279 cm, which did not represent an apparent calculation, was low-frequently chosen (4.3 %). A small group of pupils (13.3 %) had not tried to solve the item.

It seems from the distribution of the frequencies that the encoding of the problem, in contrast to the calculations themselves, was most challenging. The problem consists in both a subtractive and a multiplicative comparison situation. Generally, comparison situations are difficult, probably because not being sufficiently dealt with in teaching (Fuson, 1992).

The four alternatives given in the fourth item (M05_01), each represents the elementary operations, division, subtraction, multiplication and addition.

M05_01

There are 9 rows of chairs. There are 15 chairs in each row. Which of these gives the total number of chairs?

Ⓐ $15 \div 9$

Ⓑ $15 - 9$

Ⓒ 15×9

Ⓓ $15 + 9$

M031303

A vast majority of the pupils (82.6 %) chose the correct alternative 15 * 9. The model of the problem situation represents a typical conceptual model of the multiplication operation, which most likely explains the high solving frequency. In addition, multiplication operations like 15 * 9 are part of the attainment targets of the mathematics syllabus for this age group.

The fifth item (M05_05) is about a comparison situation.

M05_05

A man took his 3 children to a fair. Tickets cost twice as much for adults as for children. The father paid a total of 50 zeds for the 4 tickets.

How many zeds did each child's ticket cost? Show your work.

Answer: _______________

M031247

The tickets for grown-ups cost twice the price for children. Taking the cost for children as the starting point, one grown-up ticket will cost the same as two tickets for children. This gives the total cost of 50 zeds for five tickets for children. From this follows that one child ticket costs 10 zeds.

Less than one third (32.3 %) of the pupils solved this item completely. It is worth to be mentioned, however, that this relatively low frequency was higher than the average of the countries of EU /OECD (15.5 %). Several pupils had, however, difficulties in describing how they solved the item even after having encoded the problem correctly. The roots of the low solving frequency again are to be found in the comparison situations, which are known to be difficult to encode (Fuson, 1992).

The encoding of the final item (M05_07) resulted in a model that comprises two different multiplications and a consecutive addition.

M05_07

Maria has 6 red boxes. Each red box has 4 pencils inside. She also has 3 blue boxes. Each blue box has 2 pencils inside. How many pencils does Maria have altogether?

Ⓐ 6
Ⓑ 15
Ⓒ 24
Ⓓ 30

M031173

The first alternative, 6, which simply represents the addition of 2 and 4, was low-frequent (9.7 %). The second alternative, 15, which represents the addition of all the numbers involved in the item, 2 + 3 + 6 + 4, was also low-frequently chosen (6.3 %). By multiplying 6 and 4, the third alternative, 24, was arrived at. This alternative did not deviate very much from the frequencies of the other two. The last and also correct alternative, 30, is the result of the two multiplications, 6 * 4 and 2 * 3 and the addition of the two products, 24 + 6 = 30. A little less than three fourths of the pupils (72.5 %) chose this alternative.

Even though the problem situation represents a multiplicative structure, "equal-groups" and an additive conceptual combination, the solving frequency was relatively low.

8.1.5 Division

Before giving a description of the analysis of the test items on division, it seems necessary to make a distinction between the two conceptually different types of division, namely partitive and quotitive. Conceptually, partitive division is understood as the distribution of a number of objects to a limited number of parts, persons for instance. Quotitive division, on the other hand, should conceptually be understood as a number of objects being contained in a certain number of objects, boxes for instance. These two types of division are the basis for the analysis below.

The first item (M01_01) is about a constellation of cars on a parking place.

M01_01

In a car park, 762 cars were parked in 6 equal rows. How many cars were in each row?

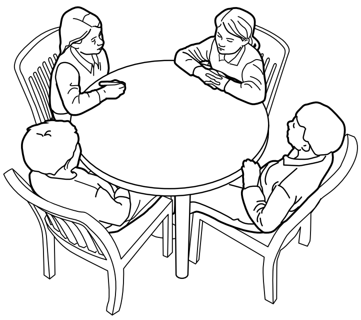
Answer: _______________

M031286

The solving frequency was low (10.1 %). This item can be solved by short division, which leads to 127 cars in each row. The explanation of the low-solving frequency is not primarily to be found in the conceptual character of the problem, since it represents a partitive division. Usually, pupils learn this early in school (Fischbein, 1985). The knowledge necessary for solving the item is also part of the attainment targets of the syllabus for grade 5.

In the second item (M02_02), there is a table with seats for four persons.

M02_02



One table can seat 4 people.

How would you find out how many tables are needed to seat 28 people?

(A) Multiply 28 by 4.

(B) Divide 28 by 4.

(C) Subtract 4 from 28.

(D) Add 4 to 28.

M041039

The four multiple-choice alternatives include the four elementary operations in order, multiplication, division, subtraction and addition. The 'multiply 28 by 4' distracter attracted most pupils (45.8 %), while the correct alternative, divide 28 by 4, was attracted by a minor group of pupils (31.9 %). The other two distracters, subtraction and addition, both had rather low frequencies (4.0 %; 12.1 %).

Quotitive division is the operation that is requested for a conceptual understanding of the problem and answers the question 'how many groups of four are contained in 28?' Since the understanding of quotitive division is at earliest developed during the later years of school, the solving frequency is relatively low (Greer, 1992).

The third item (M03_06) involves a number sentence.

M03_06

64 ÷ ■ = ■

In this number sentence, ■ stands for the same number. What number does ■ stand for?

Ⓐ 4
Ⓑ 8
Ⓒ 16
Ⓓ 32

M031255

The way of understanding the equality sign plays a decisive role in the solving of this item. A dynamic understanding of the sign makes the item more or less impossible to solve. It is the static understanding that is required (Ginsburg, 1977; Behr, Erlwanger & Nichols, 1980; Kieran, 1981; Falkner, Levi & Carpenter, 1999; Carpenter, Franke & Levi, 2003).

The multiple-choice alternatives contain multiple factors of four. The correct alternative, 8, was almost equally frequent (26.6 %) as the last distracter, 32 (24.8 %). In the EU/OECD-countries, twice as many pupils (57.0 %) on average succeeded in solving the item. The first distracter, 4, was low-frequent as was the third distracter, 16, (15.3 %). As the distracters are not possible to get at by elementary calculations, the understanding of the division sign seems to be the cause of the frequency distribution. The division sign in the item is probably understood as a subtraction sign, which gives the operation 64 – □ = □ and whose solution is 32. One fourth of the pupils (24.7 %) did not try to solve this item. The number interval seems not adapted for this age group, however, because it is comparatively high, in contrast to the next item, which had a higher solving frequency.

Layne had 32 pencils and 4 boxes for the pencils (M04_05).

M04_05

Layne had 32 pencils and 4 boxes for the pencils.
He put the same number of pencils into each box.
Which number sentence describes how many pencils he put into each box?

Ⓐ 32 + 4 = □
Ⓑ 32 – 4 = □
Ⓒ 32 × 4 = □
Ⓓ 32 ÷ 4 = □

M041281

Likewise, the four alternatives comprise the four elementary operations: addition, subtraction, multiplication and division. Approximately two thirds of the pupils (62.2 %) chose the correct alternative, 32 ÷ 4 = ☐. All together, the distracters were quite low-frequent (29.7 %). The solving frequency was, however, relatively high compared to the previous item (M02_02). This is due to the fact that the problem situation is modelled by a partitive division, which is usually learnt in earlier years compared to quotitive division (Fischbein et al., 1985; Greer, 1992). Moreover, the number interval is probably also more adapted to this age group.

Also the next item (M05_02) has a relatively high solving frequency (62.0 %).

M05_02

A piece of rope 204 cm long is cut into 4 equal pieces. What is the length of each piece?

Answer: _______________ cm

M031309

Despite the relatively high solving frequency, one third of the pupils (29.7 %) did not solve the item. Carrying out the short division is unchallenging, since the numbers are well adapted. Conceptually, the problem situation is to be understood as a partitive division. This together with the adapted number interval increased the solving frequency.

The final item (M07_05) did not have an equally high solving frequency (31.8 %) as the two items above. Chris was putting boxes on a shelf.

Yet again, the alternatives are given by the four elementary operations: subtraction, division, addition and multiplication. The most attractive distracter, 240 – 20 = ▲, was chosen by nearly one third of the pupils (29.4 %). Altogether, the other two distracters were chosen by about one fourth of the pupils (26.4 %). A minor group did not try to solve the item.

M07_05

A shelf is 240 cm long. Chris is putting boxes on the shelf. Each box takes up 20 cm of shelf space. Which of these number sentences shows how many boxes Chris can fit on the shelf? The number of boxes is shown as ▲.

Ⓐ 240 − 20 = ▲

Ⓑ 240 ÷ 20 = ▲

Ⓒ 240 + 20 = ▲

Ⓓ 240 × 20 = ▲

M031254

Since no calculation was required, the calculation itself did not hinder the pupils to solve the item. The number of times that 20 cm is possible to contain in 240 cm represents a quotitive division. Compared to partitive division, quotitive division is a conceptually more difficult operation. This fact can explain the relatively low solving frequency (Fischbein et al., 1985; Greer, 1992).

8.1.6 Fractions

Straightforward calculations with vulgar fractions are part of the attainment targets of the mathematics syllabus for this age group. By simple calculations are usually meant operations with two or three fractions with the same denominator.

The first item (M04_03) deals with equivalent fractions.

M04_03

M041069

Which fraction is equal to $\frac{2}{3}$?

(A) $\frac{3}{4}$

(B) $\frac{4}{9}$

(C) $\frac{4}{6}$

(D) $\frac{3}{2}$

The most frequently chosen alternative (64.4 %) was 3/2, which is the inverse value of 2/3. The correct alternative, 4/6, attracted only a few pupils (10.3 %) and so did the two other alternatives.

Even though equivalent fractions are part of the attainment targets of the syllabus, taking this result into consideration, the pupils do not seem familiar with equivalent fractions, which could depend on insufficient teaching in this field. Also for pupils within the EU/OECD-countries, the average solving frequency was not particularly high (20.6 %).

In the second of these items (M04_04), the encoding resulted in an addition of the two fractions with equal denominators.

M04_04

M041076

Joe spent $\frac{3}{10}$ of his money on a pen and $\frac{5}{10}$ of it on a book.

What fraction of his money did he spend?

Answer: _______________

The solving frequency was low (18.8 %). Despite the fact that simple fractions are part of the syllabus, a majority of the teachers have not taught their pupils simple fractions. In the encoding of the problem, the concept of proportion plays a key role. This concept is relatively unknown to the pupils. Besides, if the teaching has had a procedural approach, the concept of proportionality will probably not have been dealt with. In the comparative countries of EU/OECD, the average solving frequency was higher (38.5 %).

The third item (M05_03) is about division, which in Sweden is often interpreted as equivalent fractions, since division and fractions, have the same notation.

M05_03

$12 \div 3 = \blacksquare \div 2$

In this number sentence, what number does $\blacksquare$ stand for?

Ⓐ 2
Ⓑ 4
Ⓒ 6
Ⓓ 8

M031245

The correct alternative, 8, was chosen by one fourth of the pupils (18.8 %), while the most attractive distracter, 4, by half of the pupils (45.3 %). The reason for this seems to be found in the understanding of the equality sign. Dynamic understanding implies that the result of the calculation on the left side shall be written on the right side. The result of the calculation of 12/3 is 4, which means that 12/3 is seen as a division operation. Static understanding of the equality sign, on the other hand, means that the amounts on the two sides shall be the same or equivalent. Thus, the calculation of the right side shall also give four. Consequently, eight shall be divided by two. However, the two sides can also be seen as equivalent fractions 12/3 = 8/2. The average solving frequency in the EU/OECD-countries was roughly the same (21.4 %).

The final item (M07_01) is a subtraction of two fractions.

M07_01

$\frac{4}{5} - \frac{1}{5} =$

Ⓐ $\frac{3}{5}$
Ⓑ $\frac{3}{10}$
Ⓒ $\frac{3}{25}$
Ⓓ 3

The correct alternative, 3/5, was chosen by more than one third of the pupils (37.1 %). The distracter, 3, attracted nearly one fourth of the pupils (23.7 %), while the other distracters were low frequent (18.4 %) and chosen altogether by just about one fifth. A somewhat larger proportion of the pupils (20.8 %) had not tried to solve the item.

Like in this case (23.7 %), a frequent way of solving fractions by addition and subtraction is that the denominators are treated in the same way as the nominators. This gives 5 – 5, which is zero, which in turn is possible to comprehend as nothing. This is the reason for getting 3, which is the most frequent distracter, a distracter that clearly shows the misunderstanding of how fractions are to be added or subtracted.

$$\frac{4-1}{5-5}=\frac{3}{0}=\frac{3}{\text{nothing}}=3$$

This way of reasoning reflects a misunderstanding of the calculation. As mentioned above, the low solving frequency is probably due to the fact that the teaching has omitted calculations with fractions. In the other EU/OECD-countries, the average solving frequency was higher (51.2 %).

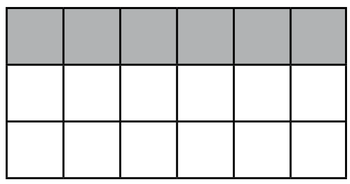
8.1.7 The Concept of Proportionality

The two main concepts that were tested are proportion and correspondence. The two concepts also showed to be mixed up.

The first item (M02_04) is about proportion. Part of a rectangle is shaded and the pupils were to determine its size. One of the distracters represents the result of a solution in which the correspondence concept is the starting point.

M02_04

What fraction of this rectangle is shaded?



Ⓐ $\frac{1}{4}$

Ⓑ $\frac{1}{3}$

Ⓒ $\frac{6}{12}$

Ⓓ $\frac{2}{3}$

M041006

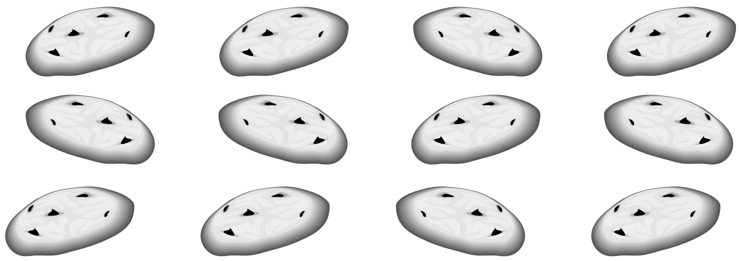
The correct alternative is 1/3, which was chosen by almost two thirds of the pupils (61.2 %). The second most frequent distracter 6/12 was chosen by nearly one third (30.0 %). This alternative can seem tricky, as it illustrates a correspondence between 6 parts and 12 parts. In this way, it has a conceptual meaning. The other two alternatives have no apparent conceptual meaning and were low-frequent (3.0 %; 5.8 %).

Proportion is about part and wholeness. In this case, the part consists of 6 squares and the wholeness of 18 squares. If one disregards the vertical division and only observes the horizontal division, there will be 1 rectangle, the part, and 3 rectangles, the wholeness. This gives 1/3.

What seems difficult with this problem is the possibility to mix up proportion and correspondence, which at least one third of the pupils did.

The second item (M04_02) is also about proportion and is of a similar character as item (M02_04). Instead of squares, twelve cookies are involved.

M04_02



M041056

There are 12 cookies. Draw a circle around $\frac{1}{3}$ of the cookies.

The cookies are deceitfully grouped in four columns and three rows. The pupils were asked to draw a circle around the proportion of 1/3 of the cookies, which gives 4 of the wholeness, 12. Somewhat less than half of the pupils managed to solve the item (46.5 %). The most frequent mistake was a circle drawn around three cookies (29.3 %). This, in contrast, represents a correspondence, where the part 3 corresponds to the part 9. Accordingly, the cookies in one column correspond to the cookies in the three other columns.

In these last two test items, the most frequent mistake seems to be that the two concepts of proportion and correspondence are mixed up (Bentley, 2008a).

The item (M01_05) involves a proportional increase.

M01_05

The temperature at 7 a.m. one morning was 12°C. It increased by 2°C every hour until it reached 20°C at 11 a.m. What was the temperature at 9 a.m.?

Ⓐ 14°C

Ⓑ 15°C

Ⓒ 16°C

Ⓓ 17°C

M031335

The most frequently (69.7 %) chosen alternative, 16 degrees, can be reached by noticing that it is two hours between 07.00 am and 09.00 am. If the temperature rises two degrees each hour, it will have risen four degrees by 09.00 am. Starting with 12 degrees and then adding 4 degrees gives the correct temperature 16 degrees. Thus, at 09.00 a.m., the temperature is 16 degrees.

The first alternative, 14 degrees, was chosen by a minor group of pupils (14.8 %) and seems to represent an additive increase. This means that only two degrees (not per hour) are added to the original temperature, 12 degrees, and results in 14 degrees.

M03_02

The next item (M03_02) is about two persons running different distances.

Two boys went running. For every 2 km that Fred ran, Alan ran 3 km.
Fred ran 6 km. How far did Alan run?

Answer: _______________ km

M031285

In table 10, the categories of the pupils' ways of reasoning are displayed. Nearly two thirds of the pupils (55.3 %) performed a correct encoding of the problem. This implies, first and foremost, that the conceptual situation has been identified. The second sentence of the item expresses a correspondence between Fred's 2 km and Alan's 3 km. This multiplicative comparison can be expressed as Alan ran 1.5 times the distance ran by Fred, which gives 1.5 * 2 = 3. If Fred ran 6 km, then Alan ran 1.5 * 6 km = 9 km.

Table 10 The Categories of the Ways of Reasoning in the Pupils' Solving of the 'Correspondence Problem', M03_02, Grade 4, n = 452

Categories	Frequency*	Relative Frequency (%)	Typical Answer
Correct Encoding and Correct Calculation	205	55.3	9
Correct Encoding but Incorrect Calculation	16	3.5	6
Incorrect Encoding but Correct Calculation Including Additive Reasoning	105	23.2	7
Incorrect Encoding and Incorrect Calculation	59	13.1	-
Nor Performed	67	14.8	-
Total	**452**	**100.0**	

*not weighted frequencies

The most frequent mistake (12.4 %) was the comprehension of the correspondence as an additive comparison instead of a multiplicative comparison. Since Fred first ran 2 km and then 4 km, the additive increase is 4 km, which is added to Alan's distance, 3 km, and results in 7 km.

A few pupils (8.2 %) only seemed to copy the text and answered that Allan ran 3 km. Surprisingly enough, a relatively large number of pupils solved the item even though the problem situation represents both a correspondence and a multiplicative comparison.

8.1.8 Summary and Conclusions

The solving frequencies of the pupils' understanding of the number concept indicate that one fourth have to develop their understanding further in order to understand the concept more thoroughly. Also, the reversing of digits in numbers has to be treated.

From the analyses were seen that the main difficulty about addition is the encoding of the problem situations representing a comparison and not the calculations per se.

Concerning subtraction, the two major difficulties are the encoding of the problem and the operation with trading.

In multiplication, the encoding of the test problems showed to be the great difficulty, even when some of the multiplication situations represented an equal-group-situation. Also the multiplication operation was hard to process. In algorithms, the concatenated structures were the core difficulty and led to the multiplication of tens and ones separately. Calculation mistakes were frequent, which indicate that the level of retrieved multiplication number facts was not fully reached.

The difference between the two conceptually different types of division, partitive and quotitive, was not understood and caused big difficulties. The encoding of quotitive divisions gave low solving frequency in contrast to the encoding of partitive divisions.

In spite of the fact that simple fractions are part of the attainment targets of the syllabus, fractions have not been taught judging from the test outcome. Less than fifty per cent of the pupils were able to solve items on simple fraction.

Proportionality is a tricky concept. The two sub-concepts, proportion and correspondence, were mixed up by the pupils, who did not understand the comprehensive concept proportionality and let alone its two sub-concepts. Proportion represents a part of the whole relation, while correspondence signifies the relation between parts within the whole. Also, a multiplicative increase was mixed up with an additive increase.

Although the specific difficulties attached to various operations were of a considerable amount, the encoding of text problems was more crucial, which was evident from a group of pupils who were not able to solve any such items, where encoding of the problem was necessary.

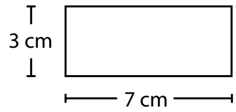
8.2 Geometry

The concepts of perimeter and area are often mixed up by the pupils. These circumstances will therefore be considered in the first section, which deals with a test item on perimeter. In the second section, pupils' answers and solutions on area are analysed. Knowledge of a number of geometrical shapes is of significant importance in order to classify their conceptual attributes. Therefore, there are several test items on triangles and rectangles, the analyses of which will be treated in the two consecutive sections. The development of pupils' spatial ability is decisive for the understanding of two-dimensional representations of three-dimensional objects. This will be described in the fifth section. Basic depictions, like rotation, translation and symmetry are explained in the corresponding section. Before the sections of summary and conclusions, the solutions of a measurement item, namely estimation, are accounted for.

8.2.1 The Concept of Perimeter

In the item, (M02_07), the perimeter is to be determined. The context represents a procedural approach, since the ways the measures are given admit a more or less direct application of the perimeter formula. Besides the correct answer, 20 cm, the measure of the area of the rectangle, 21cm, is another multiple-choice alternative but with an incorrect unit.

M02_07



What is the perimeter of this rectangle?

Ⓐ 7 cm
Ⓑ 10 cm
Ⓒ 20 cm
Ⓓ 21 cm

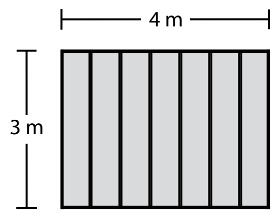
M041330

Nearly three fourths of the pupils (73.5 %) chose the correct alternative, while the distracter representing the measure of the area of the rectangle was low-frequent (3.3 %). The most frequent distracter (14.2 %) was the addition of the length and the breadth, which gives 10 cm. To give the measure of the area of a rectangle by adding its two sides is described by Clements and Stephan (2003) as a premature way of describing the size of a geometrical figure by means of a linear measure.

8.2.2 The Concept of Area

In the test item, (M04_08), Patrick is about to paint a fence.

M04_08



Patrick is painting one side of a fence. The fence is 4 meters long and 3 meters high. What is the area that Patrick has to paint?

Ⓐ 4 square meters
Ⓑ 7 square meters
Ⓒ 12 square meters
Ⓓ 14 square meters

M041152

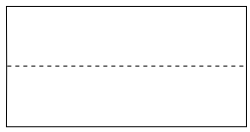
Among the alternatives is not only the measure of the perimeter, 14, although with an incorrect unit, but also the correct answer for the area, 12 square metres. The most frequent alternative (43.3 %), 7, represents the measurement of half of the perimeter, which is the sum of the two given sides. As mentioned before, younger pupils often use linear measures to get the size of a rectangle, like adding its length and breadth to get its sum (Clements & Stephan, 2003). The high frequency of this alternative reflects the shortage of challenging linear measures of two-dimensional qualities in teaching.

The correct alternative was chosen by roughly one fifth of the pupils (20.6 %) as was the measure of the perimeter (18.6 %). The first distracter, 4 square metres, was low-frequent (8.5 %).

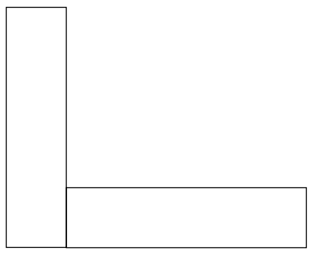
The item (M05_06) casts light upon the additive character of the concept of area and its conservation. Jill cuts a rectangle into two equal parts.

M05_06

Jill had a rectangular piece of paper.



She cut her paper along the dotted line and made an **L** shape like this.



Which of these statements is true?

Ⓐ The area of the **L** shape is greater than the area of the rectangle.

Ⓑ The area of the **L** shape is equal to the area of the rectangle.

Ⓒ The area of the **L** shape is less than the area of the rectangle.

Ⓓ You cannot work out which area is greater without measuring.

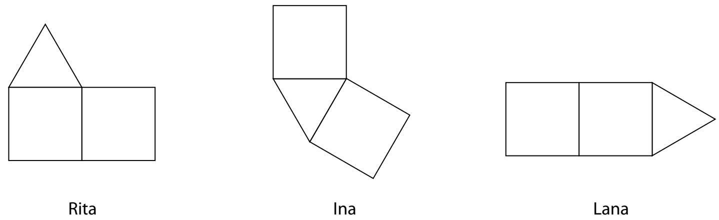
M031219

As can be seen in the item, the alternatives comprise statements about the area of the re-shaped figure. A little less than two fifths of the pupils (39.7 %) chose the correct alternative, namely the area of the L-shape is equal to the area of the rectangle. The most frequent distracter (24.0 %) was the statement that the area of the L-shape is larger than the original rectangle. Even the last distracter, the impossibility of working out which area is larger, was frequently chosen by the pupils (20.9 %).

Considering the results, a majority of the pupils lack sufficient experiences of the area concept from their teaching. This is the reason for not being able to determine if the size of the area changes when the rectangle is divided into parts and put into another shape. Knowledge of a constant size of an area of a geometrical figure if cut into pieces and put together into another shape, is based on the principle of conservation of an area (Kordaki & Potari, 1997; Piaget, Inhelder & Sheminska, 1981).

Also in the following item (M07_06), the area conservation principle is tested. In contrast to the previous item, the starting point is three distinguishable tiles that are put together in three different ways.

M07_06



Rita, Ina, and Lana take turns arranging 3 tiles. Each arranges the tiles in a different shape as shown above. Which of the following is true about the area of the shapes?

Ⓐ Rita's shape has a greater area than the others.

Ⓑ Ina's shape has a greater area than the others.

Ⓒ Lana's shape has a greater area than the others.

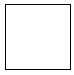
Ⓓ All of the shapes have the same area.

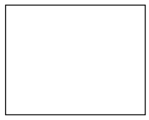
M031038

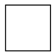
The four alternatives comprise statements about the areas of the three different shapes put together by three tiles each. The correct alternative, all of the shapes have the same area, was given by a little less than three fourths of the pupils (71.5 %). Alternative B was the most frequent distracter (13.3 %), Ina's shape has a larger area than the others. A majority of the pupils realized that the size of the areas of the three different shapes remained the same, even if the three tiles were arranged into various shapes.

The character of the area concept is also exposed in the next test item, (M07_09). The basic principle of measuring an area by covering it is tested. Julio's tile, the correct alternative, was chosen by less than four fifths of the pupils (78.5 %). Pierre's tile, however, was a frequent distracter (9.7 %). Because his tile has the least area, the number needed to cover the floor was, therefore, the largest. There can also have been difficulties in encoding the problem.

M07_09

Chris has lots of tiles like this: 

Julio has lots of tiles like this: 

Pierre has lots of tiles like this: 

Ben has lots of tiles like this: 

Who would need the least number of tiles to cover a classroom floor with his tiles?

A Chris
B Julio
C Pierre
D Ben

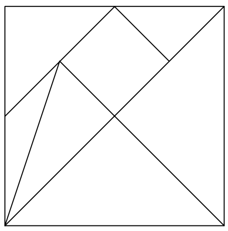
M031006

8.2.3 The Concept of Triangle

The concept of triangle is dealt with in the five consecutive items. In the first item, (M01_09), two triangles of the same form and size, are to be chosen.

M01_09

The square is cut into 7 pieces. Put an X on each of the 2 triangles that are the same size and shape.



M031271

One of the two triangles of the same form and size constitutes one fourth of the whole area of the square. In addition, the two triangles are right-angled. A large majority of the pupils (86.3 %) solved the item successfully. So, classifications of the attributes of geometrical shapes according to form and size do not seem to cause the pupils any problems.

The next item, (M02_08), comprises the four sub-items A, B, C and D. Because of the creative character of the items, conceptual understanding is not of any particular importance.

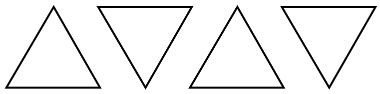
M02_08

Geometry Tiles
Instruction:

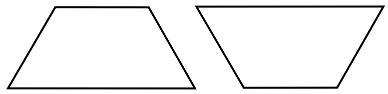
For this item, you have been given a piece of cardboard with 6 tiles like the ones shown below. Take the piece of cardboard, and punch out the 6 tiles.

If you do not have the piece of cardboard, raise your hand.

4 Triangle Tiles

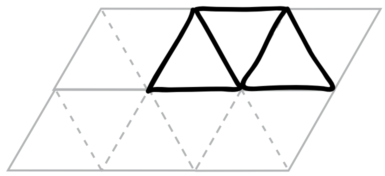


2 Trapezoid Tiles



These tiles can be used to make new figures. One problem has been done for you

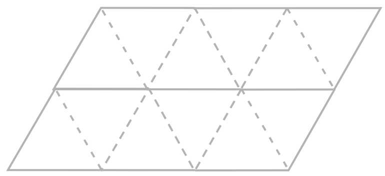
USE:	3 Triangle Tiles
MAKE:	A trapezoid
SHOW:	Draw it on the grid.



Now try these problems.

A.

USE:	1 Triangle Tile and 1 Trapezoid Tile
MAKE:	A 4-sided figure
SHOW:	Draw it on the grid.



M041300_1

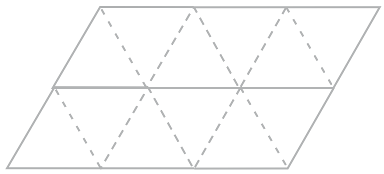
The text "A 4-sided figure" in item A, clearly indicates a figure of four sides. The solving frequency was relatively high (59.6 %).

In item B, the information goes: "Make a 6-sided figure." It is only required to construct a figure with six sides with the help of two trapezoid tiles. Somewhat less than three fifths of the pupils (59.3 %) solved the item correctly.

M02_08

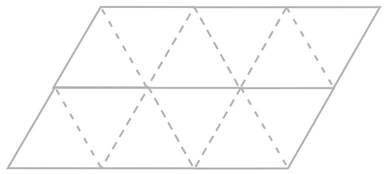
B.

USE: 2 Trapezoid Tiles
MAKE: A 6-sided figure
SHOW: Draw it on the grid.



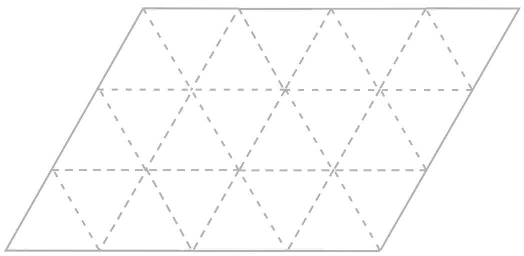
C.

USE: 2 Trapezoid Tiles
MAKE: A 6-sided figure that is not the same shape as the one you made in Part B
SHOW: Draw it on the grid.



D.

USE: 2 Triangle Tiles and 1 Trapezoid Tile
MAKE: A 7-sided figure
SHOW: Draw it on the grid.



M041300_2

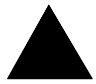
In item C, the pupils are again requested to construct a 6-sided figure but not of the same form as in item B. This seems to be somewhat more difficult and less than half of the pupils (43.9 %) were successful.

To construct a 7-sided figure by using two triangle tiles and one trapezoid tile, as is demanded in item D, was also difficult. Less than half of the pupils (48.2 %) managed to do so.

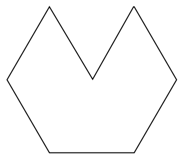
The pupils' creative ability, which is requested in order to put different geometrical shapes into new constellations, is tested in the last two mentioned test items. A prerequisite is of course knowledge of and experience from 4-sided, six-sided and seven-sided shapes. Practicing to re-shape various geometrical shapes and hereby experience the additive character of the area concept facilitates the solving process.

Item (M03_07) comprises triangles, the additive character of the area concept and the principle of measurement.

M03_07



How many triangular tiles like this are needed to cover the figure below?



Answer:_______________

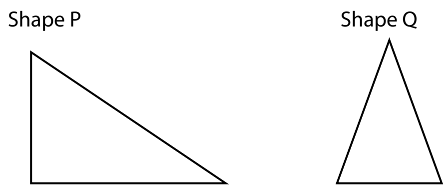
M031041

More than two fifths of the pupils (43.7 %) solved the problem and arrived at 5 triangular tiles. The ability to divide the shape at the bottom into appropriate parts is probably decisive for a successful outcome of the solving process. Moreover, the ability to master the principle of measurement based on quotitive division, is a requisite. Also analytic experiences of situations similar to the one described in the Japanese lesson in the TIMSS video study are advantageous (Stigler & Hiebert, 1999).

In item (M04_09), classifications of shapes and their attributes re-appear.

M04_09

Two shapes are shown below. Describe one way they are the same and one way they are different.



A. Same

B. Different

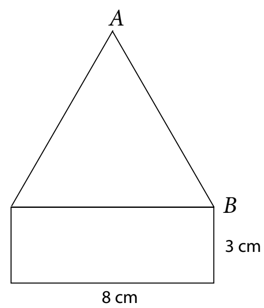
M041258

A. Same: A rather large group of the pupils (65.8 %) was able to find features common to the two triangles: both have three corners or three sides.

B. Different: A less large group (43.2 %) was able to describe features that differ: the left triangle has a right angle in contrast to the right. Practising to classify and compare different geometrical shapes in class improves pupils' ability to be successful in tasks of these kinds. Considering these circumstances, excessive independent work will not be of the best use to the pupils.

The properties of triangles are also the core issue in the next item (M05_08).

M05_08



The figure above is made from a rectangle and a triangle with three equal sides.
What is the length, in centimeters, of side *AB* ?

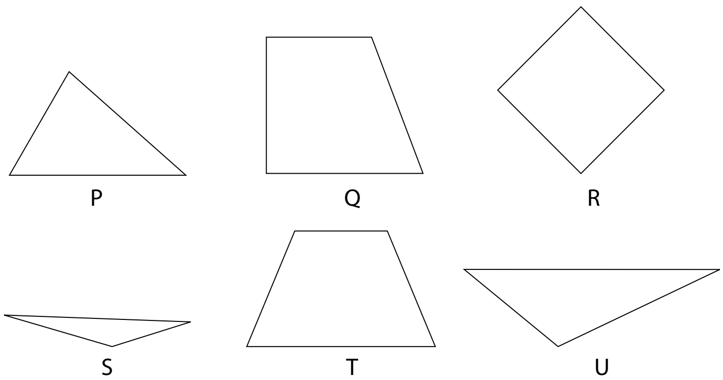
Ⓐ 8
Ⓑ 9
Ⓒ 10
Ⓓ 11

M031085

The correct alternative, 8 cm, was figured out by two thirds of the pupils (63.5 %). By stating that the two parallel sides are of equal size and that the size of one of them is given, 8 cm, it is easy to see that the other side is also 8 cm. Consequently, the base of the triangle is 8 cm, since the triangle stands on of one of the sides of the rectangle. Knowing from the text that the three sides of the triangle are of equal length, the side AB is 8 cm. A piece of information that is not necessary for solving the item is the length of the short side of the rectangle, 3 cm. One of the distracters, 11 cm, is arrived at by adding 3 cm to 8 cm (10.3 %). The most frequent distracter (12.8 %), 9 cm, is not possible, however, to get by simple arithmetic operations.

In the next item (M07_10), the pupils are to pick out the shapes that are triangles.

M07_10



List the letters of all the shapes that are triangles.

Answer: ______________________________

M031330

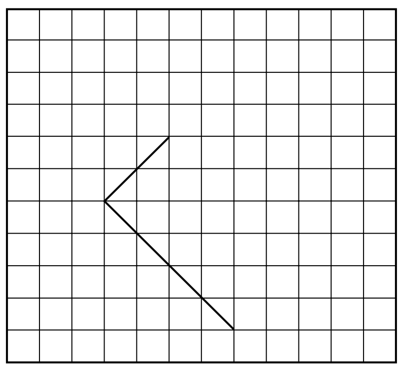
Most pupils (86.1 %) succeeded in solving this item and arrived at the correct answer P, S and U. Because triangles always have three corners, they are easy to distinguish from other figures, especially when all the others are 4-cornered. Accordingly, the high solving frequency is hardly surprising.

8.2.4 The Concept of the Rectangle

The concept of rectangle has partly been tested in the previous items. In the item below, (M04_07), this concept is particularly tested.

M04_07

Here are two sides of a rectangle. Draw the other two sides.



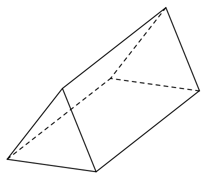
M041146

Two of the right-angled sides of a rectangle are given on a grid, which helped when drawing its other two sides. Approximately two thirds of the pupils (66.2 %) solved the item of which the main difficulty seemed to be the construction as such.

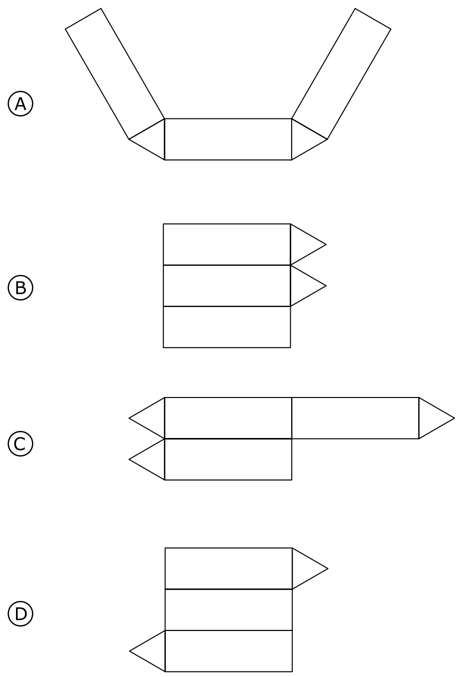
8.2.5 Two-Dimensional Representations of Three-Dimensional Objects

In order to be able to interpret a two-dimensional representation of a three-dimensional object, a well developed spatial ability is required. This ability is tested in the item (M07_11) below.

M07_11



Which of these could be folded to make a shape like the 3-D figure above?



M031351

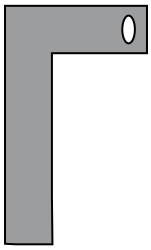
The correct alternative D was chosen by hardly half of the pupils (49.0 %). Out of the distracters, alternative A was the most frequent (22.3 %).

Spatial ability needs to be practiced to be developed. There are international computer animation projects, which have proved to be very successful (Ben-Chaim, Lappan & Houang, 1988). Although having practised their spatial ability by means of computer animation, even talented pupils showed to have difficulties in imaging two-dimensional representations out of three-dimensional objects (Ryu, Chong & Song, 2007).

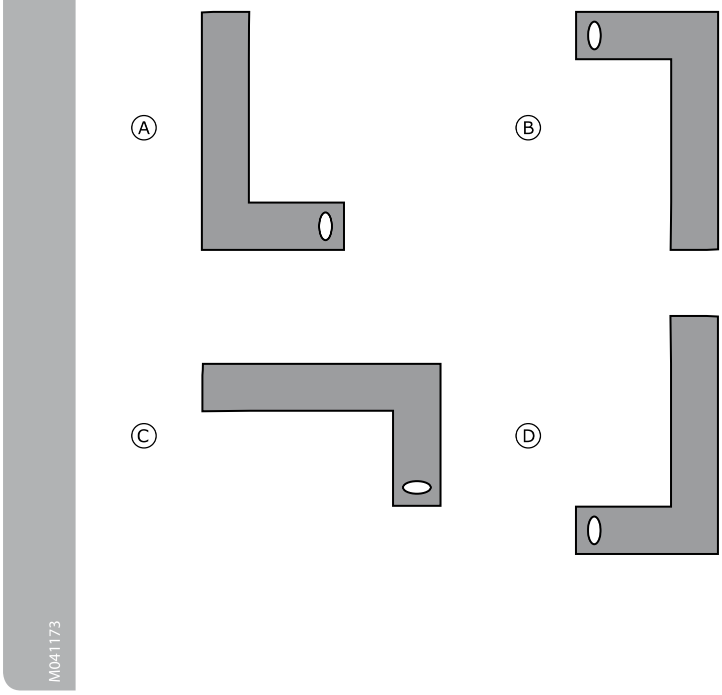
8.2.6 Conformed Depictions of Geometrical Shapes

The depictions referred to here are rotations and reflections. Rotation is dealt with in the first item (M02_09).

M02_09



The shape above is rotated by 90° clockwise. Which shape is the result?



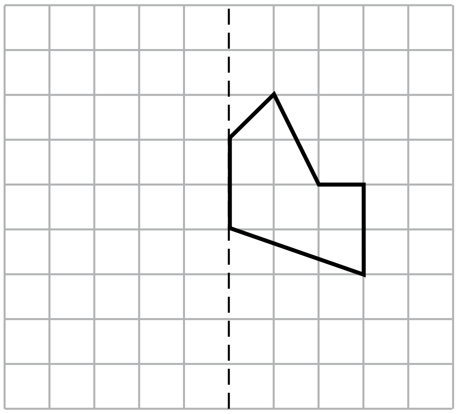
Rotation is a concept that is more focused in the syllabus than the concept of reflection, since rotation is part of the most frequent conceptual model of the angle concept. Despite this fact, only somewhat more than one fourth of the pupils (27.4 %) picked out the correct shape, C. Approximately one fifth (21.9 %) chose the shape D, which is rotated 180 degrees clockwise. D is also the most frequent distracter, closely followed by B (20.1 %), which is arrived at by being vertically reflected. About one fourth of the pupils (24.8 %) did not try to solve the item.

In the most frequent model of the angle concept, rotation is included. According to international research, the angle concept is generally difficult to understand and is therefore part of the teaching not until the later grades (Foxman & Ruddock, 1983; Mitchelmore & White, 1998). This circumstance can explain the relatively low solving frequency.

In the next item (M03_09) pupils' knowledge of reflections is tested.

M03_09

On the grid below, draw the reflection of the shape in the dotted line of symmetry.



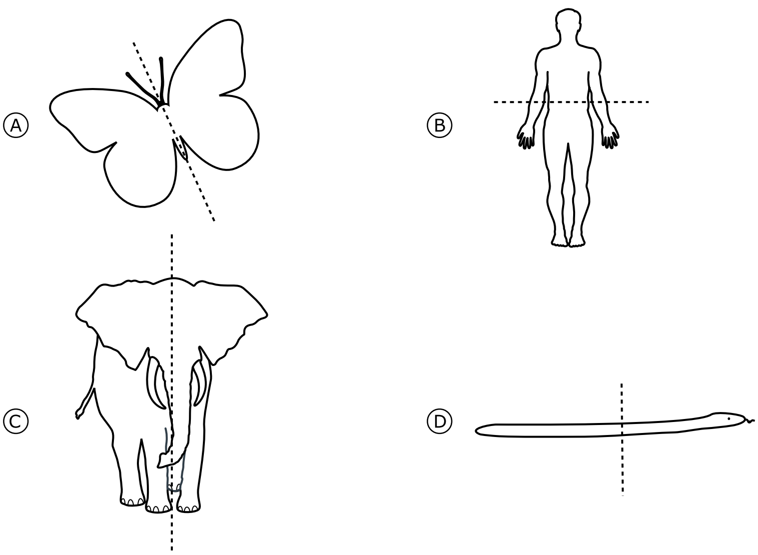
M031274

Hardly three-fifths of the pupils (57.5 %) solved the item correct. This could be due to the fact that several teachers let the pupils fold papers with ink on one side and in this way a reflection is arrived at on the other side. These kinds of experiences could have facilitated pupils' possibilities to reach a correct solution.

The exercises proposed in the analysis of the latter item could have had the same effect for a successful solution of this item (M04_06).

M04_06

In which of these drawings is the dotted line a line of symmetry?



M041164

Somewhat below half of the pupils (49.5 %) chose alternative a) the butterfly, as was the correct choice. The other alternatives got approximately each the same solving frequency (about 10 %). The c) alternative could be deceitful, since the two-dimensional projection of the animal is referred to in the item and not the three-dimensional representation, in which the spotted line could be perceived as if it shows the bilateral symmetry of the animal.

8.2.7 Measurement

The section measurement is represented by an estimation of the length of a tree (M04_10).

M04_10

The man in the picture is 2 meters tall. Estimate the height of the tree.

Ⓐ 4 meters
Ⓑ 6 meters
Ⓒ 8 meters
Ⓓ 10 meters

M041131

Less than two-fifths of the pupils (38.6 %) chose the correct alternative, 8 metres. The most frequent distracter (34.7 %) was 6 metres. The man was two metres and the estimation of the length of the tree could be done by determine how many times longer is the tree compared to the man. Thus the problem situation represents a multiplicative comparison in which the principle of measurement and quotitive division are the key phenomena. This makes the item more complicated. The tree is four times the length of the man. And four times two makes eight. The relatively difficult encoding of the multiplicative problem situation can thus explain the low solving frequency (Fischbein et al., 1985; Greer, 1992).

8.2.8 Summary and Conclusions

A large group of pupils showed to master the calculation of the perimeter of a rectangle by a procedural approach.

The area concept was possible to mix up by the perimeter concept in different ways. In one item a large group of pupils utilised half of the perimeter as a measure of the size of a rectangle. In another item a proportion of the pupils chose the correct calculation of the area and an equally large proportion the measure of the perimeter as a measure of the size of the figure.

The additive character of the area concept and its conservation were tested in several items. If a shape was divided and put together again but in another way, only a minor group of the pupils realised that the area remained indifferent. If in contrast the starting point was three different tiles, which would be put together to new shapes in a number of different ways, then a larger group realised that the areas of the composite shapes were equally large.

In one item in which the principle of measurement was focused, also the additive character of the area concept was involved. A majority of the pupils mastered the difficulty of this item. A number of triangular tiles were used to cover a shape that could be divided into triangles. This was also an application of the measurement principle. Less than half of the pupils solved this item, which was probable due to the difficulty linked to the division of the shape into triangles.

The conceptual attributes of triangles were manifested in different ways in some of the items. It seems that most pupils have understood congruence that is the same size and form, while only a minor group understood none-congruence. Most pupils managed to recognize triangles, but they did not seem familiar with triangles with special attributes. Several items of creative character, like constructing geometrical shapes, most pupils solved. But when more complex shapes were to be constructed the solving frequency decreased.

Approximately two-thirds of the pupils solved the item, in which the pupils were encouraged to construct a rectangle when two sides were given. The main difficulties seemed to be the construction per se.

The spatial ability of the pupils was tested by means of two-dimensional representations of three-dimensional objects. In the item the pupils were requested to fold different pieces of paper into a three-dimensional object. It is a known fact that spatial ability is developed relatively late during the years of schooling, a fact that was reflected in the solving frequency.

Rotations and reflections are depictions that are part of the geometry syllabus. In spite of the fact that rotations are a key part in the most frequent model of the angle concept, only somewhat more than one-fourth of the pupils managed to solve this item. Three-fifths of the pupils succeeded to carry out a reflection in a line. In contrast, less than half of the pupils seemed to be familiar with the concept of symmetry.

Estimation of the height of a tree, which represents a multiplicative comparison, in which the principle of measurement and quotitive division are focused, was also a challenge to the pupils. Less than two-fifths solved the item.

The Results – Pupils' Mathematical Knowledge, Grade Eight

9. The Results – Pupils' Mathematical Knowledge, Grade Eight

The results of the grade eight pupils' achievements in mathematics will be analysed and described. The results of their achievements showed to be below the EU/OECD-average. Algebra and geometry are the two fields within mathematics that will be dealt with. The influence of the procedural contra conceptual teaching on pupils' possibilities to solve the items will finally be accounted for.

9.1 Algebra

The testing of algebra comprises equations, expressions, functions and equations with two unknowns, formulas, the Cartesian plane and inequality, which all will be exemplified, described and analysed. After that, first a summary will be given and then conclusions from the results. As will be seen in the consecutive sections, some of the test items are not part of the attainment targets of the Swedish mathematical syllabus. The procedural teaching approaches that are reflected in the analyses of the results will also be discussed.

9.1.1 Equations

Three test items concern equations. In one of the items (M04_06), the total freight cost for an object was to be calculated by means of a formula, in which the independent variable was the weight in grams and the dependent the cost in zed.

M04_06

In Zedland, total shipping charges to ship an item are given by the equation $y = 4x + 30$, where x is the weight in grams and y is the cost in zeds. If you have 150 zeds, how many grams can you ship?

(A) 630

(B) 150

(C) 120

(D) 30

M042267

The correct alternative, 30 grams, was chosen by less than one fourth of the pupils (23.2 %). Had the text problem been correctly encoded, it would have led to an elementary equation, $150 = 4x + 30$. A majority of the pupils, over 70 %, did not succeed in solving it. Perhaps, pupils are used to small numbers in equations, where the solution is possible to find out by guessing. The results indicate that the pupils' knowledge of structural algebra is insufficient to say the least.

Also the two different ways of understanding the equality sign, dynamic and static, influenced the pupils' possibilities to solve the item. The dynamic understanding, which is known to affect the solving of equations negatively, implies that something that is calculated on the left side shall be written on the right side. Understanding the equality sign like this, makes the equation almost impossible to solve, because 150 does not become $4x + 30$. Instead, the static way of understanding the equality sign is necessary for being able to solve the equation. The static understanding means that it is the same value on each side of the sign. This makes the equation conceptually understandable and accordingly possible to solve. There exists a certain value on *x* that gives the same number of zeds on each side of the equality sign (Ginsburg, 1977; Behr, Erlwanger & Nichols, 1980; Kieran, 1981; Falkner, Levi & Carpenter, 1999; Carpenter, Franke & Levi, 2003).

Alternative A represents a mix-up of *x* and *y*. Instead of substituting *y* by 150, *x* has been replaced by 150, which leads to the equation $y = 4 * 150 + 30$, which gives $y = 630$. This alternative is almost equally frequent (20.6 %) as the correct one (23.2 %). Since, on the whole, each alternative has about the same frequency, guessing seems to have played an influential role, which reduces the basis for conclusions.

In the second item (M02_08), Joe is going to buy pens and pencils. The problem situation represents an additive comparison.

M02_08

M042263

Joe knows that a pen costs 1 zed more than a pencil.
His friend bought 2 pens and 3 pencils for 17 zeds.
How many zeds will Joe need to buy 1 pen and 2 pencils?

Show your work.

The pupils are expected to solve the item by means of an equation. About one third (33.8 %) succeeded in solving the item correctly. Comparing this with the EU/OECD-countries, their average solving frequency was lower (27.8 %). If one pencil costs *x* zeds, one pen will cost $(x + 1)$ zeds. The further encoding of the problem therefore led to the mathematical model $2(x + 1) + 3x = 17$, which makes $x = 3$. Since one pencil costs 3 zeds, then one pen costs 4 zeds. Consequently, Joe needs 10 zeds to buy 1 pen and 2 pencils.

The main difficulty seems to be the encoding process. There was, however, a group of pupils who solved the item without taking advantage of an equation.

The third (M07_05) of the items on equations required the knowledge of the structural algebra.

M07_05

$3(2x - 1) + 2x = 21$

What is the value of x?

Ⓐ -3

Ⓑ $-\frac{11}{4}$

Ⓒ $\frac{11}{4}$

Ⓓ 3

M032540

Also in this case, four alternatives were possible to choose among. The correct alternative 3 was chosen by about half of the pupils (50.6 %). The most frequent distracter 11/4 was possible to reach by applying the distributive law incorrectly (22.1 %), a law that some pupils did not seem to have fully internalized. By applying the distributive law, arithmetic algebra has been abandoned.

In these last two equations, the variable x represents an unknown specific number. When, in the following, expressions are to be simplified, the variables represent general numbers.

9.1.2 Expressions

The four consecutive items deal with simplifications of expressions. In item (M02_06), the pupils were asked to choose which alternative out of four is equivalent to $4x - x + 7y - 2y$.

M02_06

Which is equivalent to $4x - x + 7y - 2y$?

Ⓐ 9

Ⓑ $9xy$

Ⓒ $4 + 5y$

Ⓓ $3x + 5y$

M042199

The correct result, $3x + 5y$, was selected by almost two thirds of the pupils (61.0 %). This was probably due to the fact that the conceptual model, termed the Object Model, was directly applicable to the test item. Simplifying $4x - x$ to 4, as a number of pupils did, can depend on an additive understanding of $4x$ as 4 and x (18.5 %). The understanding of a variable typical of the Non-Symbolic Representation Category results in the distracter, 9, (4.5 %). In this way of understanding a variable, only the coefficients are added, 4 + 7 – 2 = 9, while the variables are ignored.

Item (M04_07) is the second on simplifications of expressions.

M04_07

Which of these is equal to $2(x + y) - (2x - y)$?

Ⓐ $3y$

Ⓑ y

Ⓒ $4x + 3y$

Ⓓ $4x + 2y$

M042239

The correct answer, $3y$, was low-frequently chosen (18.9 %). By not changing the sign in the negative second parenthesis, the distracter, y, (27.8 %) is arrived at. D is the most frequent alternative (32.5 %) and is come to by multiplying the first parenthesis incorrectly. Since only x and not y is multiplied by 2, $2(x + y)$ becomes $2x + y$. Moreover, the two negative signs, one before and one within the second parenthesis, were changed into positive signs and the result was $4x + 2y$. In distracter C (13.5 %), the first parenthesis is correctly multiplied by 2, whereas the two consecutive negative signs are incorrectly changed into positive signs and therefore, $4x + 3y$ was arrived at.

The rather low solving frequency (18.9 %) clearly shows that the pupils do not master to operate with parentheses, which is an essential part of the structural algebra.

In item (M05_02), $2a^2 * 3a$, another kind of simplifications of expressions is tested.

M05_02

$2a^2 \times 3a =$

Ⓐ $5a^2$

Ⓑ $5a^3$

Ⓒ $6a^2$

Ⓓ $6a^3$

M032198

The most frequent alternative (46.4 %) was the incorrect $6a^2$. This can be obtained by excluding the exponent of $3a$ ($3a^1$) when simplifying the expression $2a^2 * 3a$ into $6 * a^{2+0} = 6a^2$. If, on the other hand, the exponent had been included, $2a^2 * 3a^1$, the addition of the exponents would have given a^3 and resulted in $6 * a^{2+1} = 6a^3$, the correct alternative, which had a relatively low solving frequency (18.0 %). Another frequent mistake was that instead of multiplying the two coefficients, 2 and 3, they were added and ended up in either of the alternatives $5a^2$ and $5a^3$, whose frequency taken together was 30.7 %. As it seems, the Object Model, which is only adapted to additive simplifications and not to multiplicative, has been applied. The two alternatives, $5a^2$ and $5a^3$, can therefore be suspected to have their roots in the additive Object Model. As reported in the research review, multiplying two bananas by three bananas has no conceptual meaning. For that reason, the Object Model is not supportive to pupils' thinking. Instead, the variable needs to be seen as a generalised number.

In item (M04_03), the value of an expression is to be calculated.

M04_03

$a = 3$ and $b = -1$.

What is the value of $2a + 3(2 - b)$?

Ⓐ 15
Ⓑ 14
Ⓒ 13
Ⓓ 9

M042082

The correct alternative 15 was chosen by a minority of the pupils (10.9 %). Distracter 9, however, which was chosen by the majority (59.3 %), was arrived at by not taking into consideration that given $b = -1$ then $-b$ will not be -1 but $+1$. Distracter, 13, was the result of not being applying the distributive law correctly to the parenthesis (14.7 %).

To calculate the value of an expression is questionable from a conceptual perspective, since functions have values and not expressions. But had it been a function instead of an expression, the function would have had two independent variables, which belongs to the domain of university studies. The context of this item may seem unfamiliar to the pupils, but the knowledge needed to solve the item is on an elementary level.

9.1.3 Functions and Equations of Two Unknowns

In the following items, the concept of function is not explicitly mentioned. The functional relationships being tested are instead equations of two unknowns, graphs and diagrams.

The table below shows a functional relationship between x and y (M05_10).

M05_10

The table below shows a relation between x and y.

x	1	2	3	4	5
y	1	3	5	7	9

Which of the following equations expresses this relation?

Ⓐ $y = x + 4$

Ⓑ $y = x + 1$

Ⓒ $y = 2x - 1$

Ⓓ $y = 3x - 2$

M032163

The question was which one of the equations expressed the relationship in the table. The most frequent alternative (32.1 %) was also the correct one, $y = 2x - 1$. More than one fourth of the pupils (26.9 %) was attracted by the distracter $y = x + 1$. A typical mistake was that the pupils did not check all the pairs of numbers but only one or a few. The distracter $y = x + 1$ covers the relationship between 2 and 3 but not the rest of the pairs. The least frequent distracter, $y = 3x - 2$, cover the first pairs of numbers 1 and 1 but not the rest (8.9 %). Distracter $y = x + 4$ only covers the pair 5 and 9. The fact that every pair of numbers in the table needs to be checked seems to be unknown to a large number of the pupils. This lack of awareness could be due to the understanding of the variable as a specific unknown number. If the variable is seen as a specific unknown number, it will be impossible for the pupils to understand that the variable can take several values.

It is also worth noticing that one fifth of the pupils (20.0 %) did not try to solve the item.

The next item (M07_04) also deals with a relationship.

M07_04

2, 5, 11, 23, ...

Starting the pattern at 2, which of these rules would give each of the terms in the number pattern above?

Ⓐ Add 1 to the previous term and then multiply by 2.

Ⓑ Multiply the previous term by 2 and then add 1.

Ⓒ Multiply the previous term by 3 and then subtract 1.

Ⓓ Subtract 1 from the previous term and then multiply by 3.

M032273

Alternative B is the correct one and was chosen by somewhat less than two thirds of the pupils (62.3 %). The most frequent (15.9 %) distracter was C, which represents a multiplication of the previous number by three and then a subtraction by one. The second number in the sequence is 5 and can be reached in this way but not the remaining two.

Another explanation to why the pupils only checked one pair of numbers could be to over-generalize the un-ambiguity of the four elementary operations into relationships between variables. If a rule or relationship between the two numbers 2 and 5 shall result in 7, only one of the elementary operations can describe this, namely addition. The relationship is hereby fully determined and further checking is not necessary. In contrast to this understanding of the elementary operations, not only one compound operation but several can describe the relationship between 2 and 5 that equals 7. For example the two linear expressions, $6x - y$ and $9x - 2y - 1$, if applied to the numbers 2 and 5, will both result in 7. But for the other pairs of numbers, the two expressions do not have the same values. Therefore checking all pairs of numbers is necessary.

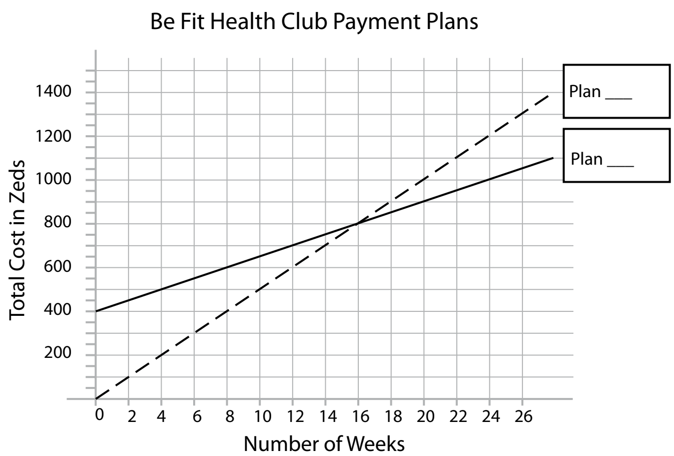
From these lines follows that the mistake of not checking whether every one of the consecutive pairs of numbers satisfies the rule can have its roots in the overgeneralization of the un-ambiguity of the elementary operations.

This test item (M07_13) is about two graphs in a Cartesian plane representing two different plans, A and B.

M07_13

The Be Fit Health Club offers two different payment plans.
Plan A has an initial fee of 400 zeds and a weekly fee of 25 zeds.
Plan B does not have an initial fee but has a weekly fee of 50 zeds.

The figure below compares the cost for Plan A and Plan B.



A. Label the line that represents the cost for Plan A, and label the line that represents the cost for Plan B.

B. At which week would you have paid the same amount for Plan A and Plan B?

C. At 24 weeks, what is the difference in total cost between the two plans?

The characteristics that make the two graphs possible to identify are given. A majority of the pupils (84.3 %) were able to spot the costs for the two plans A and B. Also the sub-item of naming the week with equal costs for the two plans was solved by a majority of the pupils (89.5 %). Finding the difference in the total cost between plan A and B, was solved by 75.8 %. This sub-item tests the ability to interpret the meaning of specific points on the two graphs.

A question natural to ask, about this test item, is why such a lot of pupils solved it. Firstly, the level of abstraction is relatively low. Generally speaking, items of a more concrete character usually facilitate the solving for many pupils. Secondly, no letter denotations were used as variables, a fact that usually improves the solving frequency. This type of items is also frequent in the various text books (Hemmi, Kirsti, written communication, 2008).

In the next item (M04_08), the pupils were to choose the point in number pair that is on the line $y = x + 2$.

M04_08

Which point is on the line $y = x + 2$?

Ⓐ (0, −2)

Ⓑ (2, −4)

Ⓒ (4, 6)

Ⓓ (6, 4)

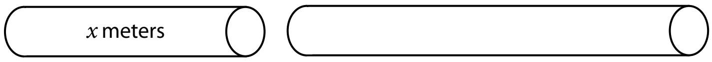
M042238

Several pupils (19.0 %) selected the incorrect distracter (0; –2). The x and y seemed, however, to be mixed up, since the point (–2; 0) satisfies the equation $y = x + 2$. Distracter (2; –4) was most frequent (34.7 %). The correct alternative (4; 6) was only chosen by barely one fourth of the pupils (23.3 %). It seemed that only a small number of the pupils were aware of the fact that the coordinates of a point have to satisfy the equation of the line. Not having this knowledge, guessing takes over. The item is not covered by the Swedish syllabus, which can partly explain the relatively low solving frequency.

9.1.4 Formulas

In the following two items, the concept of formula is focused. In item (M04_04), the pupils were requested to create a formula for expressing the length of one tube relative the length of another.

M04_04



The first pipe is x meters long. The second pipe is y times as long as the first one. How long is the second pipe?

Ⓐ xy meters

Ⓑ $x + y$ meters

Ⓒ $\frac{x}{y}$ meters

Ⓓ $\frac{y}{x}$ meters

M042088

Since the problem situation represents a multiplicative comparison, the encoding of the text problem could be the main difficulty. In spite of this, the first and also the correct alternative was relatively high-frequent (52.6 %). Approximately one fourth (24.2 %) was attracted by the second distracter '$x + y$ metres', which could be traced to an additive increase.

This item is about the basic principle of measurement, namely to examine the number of units that are required to cover a certain length. Nonetheless, the text has been interpreted somewhat differently since either of the two alternatives y/x and x/y was arrived at, neither of which seems to have any obvious conceptual meaning.

The second test item on formulas (M07_06) is about a number of jackets.

M07_06

The number of jackets that Haley has is 3 more than the number Anna has. If n is the number of jackets Haley has, how many jackets does Anna have in terms of n?

Ⓐ $n - 3$

Ⓑ $n + 3$

Ⓒ $3 - n$

Ⓓ $3n$

M032698

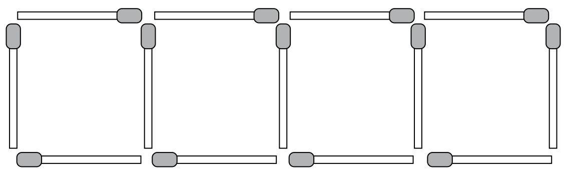
The first alternative is also the correct, $n - 3$, and was chosen by nearly two thirds of the pupils (60.9 %). The second alternative, $n + 3$, was the most frequent distracter (19.5 %), while the last two distracters were both low-frequent (17.0 %).

In this problem, the variable is to be seen as a specific unknown number, a situation which most pupils are accustomed to. This makes the problem easier to solve compared to problems where the variable needs to be seen as a generalized number. Nevertheless, there is a main difficulty, namely whether 3 shall be added to or subtracted from n. Alternative $n + 3$, however, represents a situation in which Anna has three jackets more than Haley.

Since the problem situation represents a comparison, the encoding of the problem could have caused the difficulties and not the variables and formulas as such.

In the third item (M05_03), the solving of problems by using patterns is tested. This kind of problems is included in pre-algebraic training.

M05_03



In the figure, 13 matches were used to make 4 squares in a row. What is the number of squares in a row that can be made in this way using 73 matches? Show the calculations that lead to your answer.

Answer: _______________

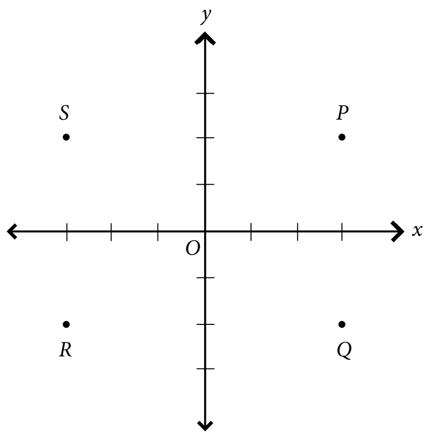
M032640

Only a minor group of pupils (16.0 %) solved the item correctly. The pattern is four matches for the first square and tree matches for each one of the next three next squares. This gives a situation of several consecutive additive changes. If the number of new squares to be added is n, the formula for the pattern will be $4 + n * 3$, since the first square have to be included. This gives the equation $4 + 3n = 73$ and $n = 23$. But since n only represents the number of new squares, the first and four-sided one has to be added. This gives the answer 24 squares. Since the additive change-situation is not considered particularly difficult to encode (Fuson, 1992), the difficulty experienced by the pupils is to be found in the non-familiarity with patterns.

9.1.5 The Cartesian plane

In a Cartesian plane, four points are marked (M02_11). The pupils have to determine which of the points have the coordinates (3; –2).

M02_11



Which of the following represents the point (3, –2) on the graph?

Ⓐ *P*

Ⓑ *Q*

Ⓒ *R*

Ⓓ *S*

M042148

Necessary to point out is that the axes are graded but not with numbers. About half of the sample of pupils made the correct choice (45.9 %), the point Q. The most frequent (22.0 %) distracter (–3; –2) was chosen by nearly one fourth of the pupils. The least frequent (7.9 %) distracter was (–3; 2) and distracter (3; 2) was chosen by approximately one sixth (15.1 %).

As the points were located symmetrically in the plane, the actual difficulty seemed to be to distinguish the meaning of the negative sign. This could, however, be tricky since the axes were not graded with numbers. Also the discrimination of the order of the numbers of the coordinates can be crucial in this kind of items. Due to the symmetrical relationship between the four points in the Cartesian plane, there was no such difficulty, since all the points have both an *x*-coordinate and a *y*-coordinate of the same absolute value.

9.1.6 Inequality

The last item (M01_04) on algebra that is included in this report concerns the solving of an inequality. However, inequalities are not part of the Swedish syllabus in mathematics for grade nine.

M01_04

$\frac{x}{3} > 8$ is equivalent to

(A) $x < 5$

(B) $x < 24$

(C) $x > \frac{8}{3}$

(D) $x > 5$

(E) $x > 24$

M022050

The inequality is solved by multiplying the two sides by the positive integer three. According to the rules of structural algebra, the inequality sign is not affected.

Out of the five multiple choice alternatives, the three distracters $x < 24$ (29.2 %), $x > 8/3$ (29.9 %) and the correct one, $x > 24$, (25.9 %) were chosen by about one third each. Distracter $x < 24$ is the opposite of the correct one, $x > 24$. That the distracter, x > 8/3 , attracted most pupils (29.9 %) could depend on an incorrect solving strategy, according to which the left side is multiplied by three and the right side is divided by three. The two distracters, $x < 5$ and $x > 5$, both involved the number five (together 7.4 %). In the first, the inequality sign means less than five and in the second more than five. Arriving at five seems to depend on the use of a subtractive approach 8 – 3 = 5.

9.1.7 The Understanding of the Variable Concept

Also, in two of the not released test items, the pupils' understanding of the variable concept was clearly exposed. As has been reported in the research review, the powerfulness of a test item determines the ways of understanding that are possible to expose. Some items even facilitate the exposure of certain ways of understanding. Furthermore, the context determines the meaning of the concept of variable.

In table 11, the categories of the pupils' exposed understanding of the concept of variable are presented.

In the specific context, the correct way of understanding belonged to the Specific Unknown Number Category, which only one fifth of the pupils exposed (19.2 %). Almost equally frequent (18.4 %) was the Digit Representation Category, in which the variable represents a digit and not a number. A characteristic feature of the understanding of the concept of variable in the Non-Symbolic Representation Category is that the letter denotation has no meaning and is therefore ignored (3.4 %). Understanding variable as in the Additive Representation Category, the variables are added instead of multiplied (16.3 %).

Table 11 The Categories of Pupils' Exposed Understanding of the Variable Concept, in the first test item, not released, n = 553

Categories	Frequency*	Relative Frequency (%)
Specific Unknown Number	106	19.2
Digit Representation	102	18.4
Non-Symbolic Representation	19	3.4
Additive Representation	90	16.3
Not Categorised	62	11.2
No Calculation Performed	174	31.5
Total	**553**	**100.0**

*not weighted frequencies

As mentioned, the character of the test item delimits the number of categories that can be exposed. In table 12, a somewhat different distribution of the understanding of the concept of variable is displayed. The solving of the item was facilitated when the variables were seen as generalised numbers. A relatively high proportion of the pupils understood it in this way (40.1 %). However, the item would also have been possible to solve, if the variables were viewed as specific unknown numbers. Yet, none of the pupils exposed this solving approach. The Specific Unknown Number Category in table 12 was linked to a mistake carried out by a number of pupils (7.4 %).

Table 12 The Categories of Pupils' Exposed Understanding of the Variable Concept, in the second test item, not released, n = 553

Categories	Frequency*	Relative Frequency (%)
Generalised Number	222	40.1
Specific Unknown Number	41	7.4
Digit Representation	2	0.3
Non-Symbolic Representation	113	20.4
Not Categorised	52	9.5
No Calculation Performed	123	22.3
Totalt	**553**	**100.0**

*not weighted frequencies

Approximately one fifth of the pupils (20.4 %) exposed the understanding of the Non-Symbolic Representation Category. The relatively low exposure of the understanding typical of the Digit Representation Category (0.3 %) was due to the low powerfulness of this specific test item.

It has been shown that a large proportion of pupils do not grasp the contextually dependent meanings of variables. Instead, several misconceptions were exposed.

9.1.8 Summary and Conclusions

First, it is necessary to conclude that the items test a rather large range of mathematical knowledge. Therefore, the analyses of the results give a rather disparate picture. Nevertheless, some common features were possible to distinguish.

Most pupils showed to have the dynamic and not the static understanding of the equality sign, which made the solving of certain equations tricky. Also the application of the Distributive law to negative parenthesis caused a lot of problems not only in the solving of equations but also when simplifying expressions.

When the Object Model was directly applicable to an item with the simplification of an expression, it resulted in a relatively high solving frequency. But in items where the Object Model was not applicable, the solving frequency was much lower. Only additive simplifications and not multiplicative can be made by this model.

Another tricky situation known is the substitution of negative numbers, which frequently gave incorrect answers. Substituting a negative variable by a negative number should result in a positive number, a fact that most of the pupils did not agree with, however.

What did not facilitate the solving of items but created false difficulties was that the terminology of functions and equations of two unknowns were mixed up. The largest difficulty, however, seemed to be the understanding of the variable concept. Several test items became extra difficult to solve, due to the fact that the meaning of the variable involved did not seem to be correctly understood. Exposed were the commonly known misconceptions that belong to the categories of Non-Symbolic Representation, Digit Representation, Additive Representation and Concrete Object Representation. Moreover, the two ways of understanding that belongs to the categories of Specific Unknown Number and Generalised Number were difficult to identify in relation to their contexts. In general, the variable was seen as a specific unknown number and not as generalised number.

As reported in the research review, pupils have a number of different ways of understanding a concept that are often contextually linked. The concept of variable is not an exception in this respect. What has been accounted for in this report is the pupils' exposed ways of understanding a number of concepts. Not necessarily do the pupils lack accurate ways of understanding the concepts but can be unaware of their applications.

Finally, it needs to be stressed that a number of items on algebra tests the mathematical knowledge that are not included in the attainment targets of the syllabus for grade 9.

9.2 Geometry

The understanding of several concepts within the field of geometry was tested. The two concepts of perimeter and area, which were often mixed up, will be dealt with both separately and in combination. One of the most frequently tested concepts, the angle concept, will be described below. The concepts of triangle and rectangle are two-dimensional objects and are represented by three test items. Two-dimensional representations of three-dimensional objects are dealt with in four test items. Finally, conformed depictions of geometric shapes will be analysed.

9.2.1 The Concept of Perimeter

The first test item (M01_05) comprises the understanding of the two concepts of perimeter and area.

M01_05

What is the perimeter of a square whose area is 100 square meters?

Answer: _______________

M022055

A crucial conceptual attribute of a square is that its sides are of equal length. This is prior-knowledge that is necessary to have acquired for being able to solve the problem. Nearly half of the sample (46.9 %) solved the item correctly. Various kinds of classified errors amounted to 24.8 %. A common mistake was that the two concepts of perimeter and area were mixed up. In international research (Clements & Stephan, 2003), there is clear evidence of this kind of mistake.

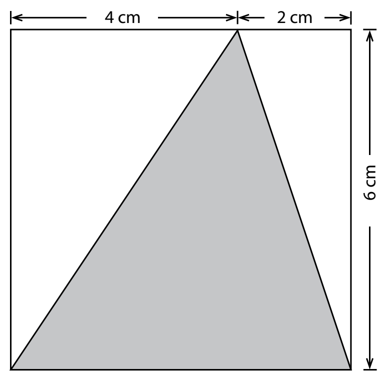
9.2.2 The Area Concept

In this section, the area concept in four test items is analysed.

The first test item (M01_12) is about the calculation of the area of a triangle that is placed in a square.

M01_12

The figure shows a shaded triangle inside a square.



What is the area of the shaded triangle?

Answer: _______________

M022243

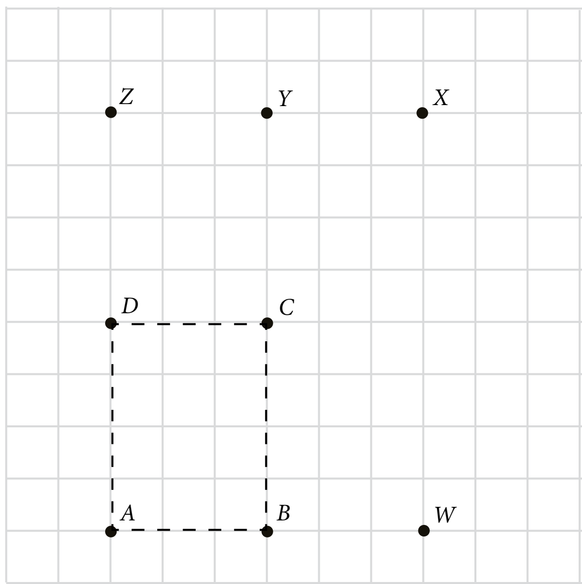
Less than half of the sample of pupils solved the problem either fully or partially (42.7 %). A number of pupils (37.5 %) did not succeed in solving it, while a minor group (19.6 %) did not try.

The pupils exposed that they did not fully understand the area concept of a triangle let alone to calculate its area. As Stigler and Hiebert concluded from their research (1999), it can be suspected that the calculation of the area of a triangle has been procedurally taught and learnt and hereby not conceptually understood. In this item, the area formula was not possible to apply directly but required a conceptual adaption. This can explain the difficulties that a relatively large group of pupils seemed to have experienced.

By means of a number of marked points the pupils shall draw a triangle that is twice the size of a given rectangle, test item (M04_11).

M04_11

Using the marked points, draw a triangle having an area TWICE that of the rectangle *ABCD*.

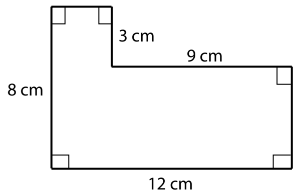


M042130

Over half of the sample (50.2 %) solved the problem correctly, while the rest did not succeed in solving it or left it out. In contrast to the item just above, the major difference is the given grid, which probably facilitated the understanding and the solving of the problem, since it could be conceptually solved by merely counting the grid squares. A significant group of pupils did not seem to have discovered this fact but yet tried to solve the item. Not using the support of the grid makes the item more difficult.

Item (M07_08) is about the calculation of the area of a hexagon, which was possible to divide into two rectangles. The area could be calculated by using the additive character of the area concept.

M07_08



What is the area, in square cm, of the figure shown above?

Ⓐ 66
Ⓑ 69
Ⓒ 81
Ⓓ 96

M032575

More than half of the pupils (51.2 %) solved the item correctly. The most frequent (21.2 %) distracter 96 is reached by multiplying the two sides 8 cm and 12 cm. The two distracters 66 and 81, which were impossible to get as a result of elementary operations, together attracted nearly one fourth of the pupils (23.8 %).

Consequently, it seems as if several pupils were unfamiliar with the additive character of the area concept (Chick & Baker, 2005). A group of pupils seemed to lack the ability to discern two rectangles in the hexagon. This difficulty is of crucial importance for advancing in the solving process.

This test item (M07_07) is about frogs in a circular pond.

M07_07

A circular pond has a radius of 10 meters. There is an average of 2 frogs per square meter in the pond. Approximately how many frogs are in the pond?

π is approximately 3.14

Ⓐ 120
Ⓑ 300
Ⓒ 600
Ⓓ 2400

M032097

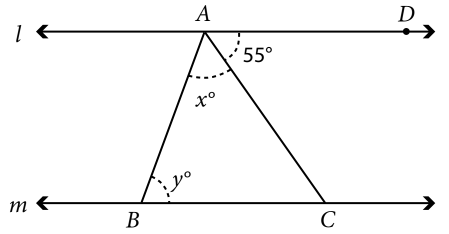
One of the optional alternatives, 120, is the measure of the perimeter of the circle multiplied by 2 (frogs per square metre). Alternative 300 is the measure of the area of the circle. Each of the two alternatives was chosen by less than one third of the pupils (32.1 %; 28.7 %). The correct alternative, 600, the measure of the area of the circle multiplied by 2, was roughly equally frequent (31.4 %). The alternative of 2400 was, however, low-frequent.

The pupils who failed seem not only to have a limited understanding of the area of a circle but also of distinguishing a circle's perimeter from its area (Clements & Stephan, 2003). It could also be concluded that the concept of proportionality plays a key role in this item, since the number of frogs was given per square metre.

9.2.3 The Concept of Angle

Eight items test the pupils' understanding of the concept of angle. In the first item (M02_10), two parallel lines were connected by a triangle, whose basis was part of one of the lines and whose top edge was placed on the other line.

M02_10



In this figure, line *l* is parallel to line *m*. The measure of angle *DAC* is 55°.
What is the value of $x + y$?

Ⓐ 55
Ⓑ 110
Ⓒ 125
Ⓓ 135

M042137

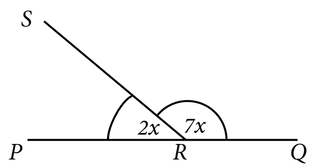
The correct alternative 125° was also the most frequent (42.3 %). However, the alternative 110° is a reasonable guess if no calculation is performed (34.4 %).

The knowledge requested to solve the problem is not only the concept of alternate angles but also the sum of the interior angles in a triangle, 180°.

A straight line is the starting point for the next item (M03_06). On the line there was a point R, from which a ray is drawn.

M03_06

In this figure *PQ* is a straight line.



What is the degree measure of angle *PRS*?

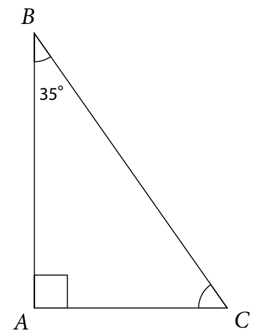
Ⓐ 10°
Ⓑ 20°
Ⓒ 40°
Ⓓ 70°
Ⓔ 140°

It is necessary for the pupils to know that the sum of the two angles, $2x$ and $7x$, is 180°. This leads to an elementary equation $2x + 7x = 180°$ and $x = 20°$. What seems to be the problem here is neither the encoding of the problem text nor the calculation but to give the correct answer, which is not the value of x but of $2x$. However, the notation $2x$ and $7x$, which expresses a proportional relationship between the sizes of the two angles, implied an increased degree of difficulty.

Alternative 20° was chosen by one sixth of the sample (13.7 %). A relative large majority picked the correct alternative 40° (59.7 %). Alternative 70° was chosen by roughly one sixth (16.7 %). The two distracters 10° and 140° were low-frequent.

In item (M03_14), a right-angled triangle with one further angle is given, 35°. The pupils are supposed to give the measure of the third angle.

M03_14



What is the measure of angle *C* in the triangle above?

Ⓐ 45°

Ⓑ 55°

Ⓒ 65°

Ⓓ 145°

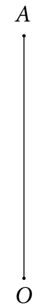
M032579

The knowledge necessary is that the sum of the interior angles in a triangle is 180°. The correct multiple choice alternative, 55°, was frequently chosen (58.3 %). About one fifth picked the most attractive distracter 45o (18.7 %). The distracter 65° had a frequency of 15.3 %. Those who failed were probably not familiar with the necessary knowledge that the sum of the interior angles of a triangle is 180°.

A segment of a line is given in item (M03_15) and the pupils are to draw a line so that one obtuse and one acute angle are created.

M03_15

Using line segment *AO* below, draw a straight line *BC* through *O* such that angle *AOB* is acute and angle *AOC* is obtuse. Label points *B* and *C*.

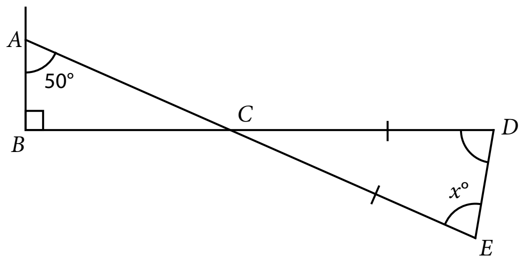


M032691

Less than one third of the pupils (30.6 %) succeeded in solving the item. In the EU/OECD-countries, however, nearly half of the pupils (46.4 %) were successful. The most probable reason for not solving the item was the pupils' unfamiliarity with the terminology of obtuse and acute angle.

In the figure of item (M04_10) the two angles ACB and DCE are vertical angles, which are always of equal size.

M04_10



In this diagram, *CD* = *CE*.
What is the value of *x*?

Ⓐ 40
Ⓑ 50
Ⓒ 60
Ⓓ 70

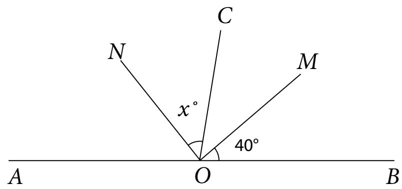
M042036

The sum of the angles in the left triangle should be used to calculate the two unknown vertical angles that per definition have the same size. Further, it is known that CE = CD. This gives an isosceles triangle with the two base angles of equal size, $x°$. The angle of $x°$ is to be calculated. Since the vertical angle in the left triangle is known to be 40°, the vertical angle in the right triangle is also 40°. By using the fact that the sum of the interior angles in a triangle is 180°, the equation becomes $2x + 40 = 180$, which makes $x = 70°$. This alternative was chosen by less than one third of the pupils (31.0 %). Of almost equal frequency was the distracter 60° (30.6 %), which could be derived from the figure by just looking at it. The distracter 50° was probably taken from the corresponding angle in the left triangle (19.3 %).

The difficulty in this item seems to be twofold, first, the knowledge of vertical angles and their equal size and second, the knowledge of an isosceles triangle and the equal size of its two base angles.

In item (M07_09), an angle is to be calculated.

M07_09



In the figure above, points *A*, *O*, and *B* lie on a line. *OM* bisects angle *BOC* and *ON* bisects angle *AOC*. What is the value of *x*?

Answer: ______________

M032414

Since the ray OM divides the angle BOC in two parts of equal size, the angle MOC is also 40°. The angle AOB is 180°, which gives the equation $x + x + 40 + 40 = 180$. Thus $x = 50$. This is also possible to arrive at by testing different values on the angles.

Less than one third of the pupils (28.0 %) solved the item correctly. It is presupposed that the pupils master the model of the angle concept. However, according to international research, only a minor group of pupils of this age has internalised this conceptual model (Foxman & Ruddock, 1983; Mitchelmore & White, 1998).

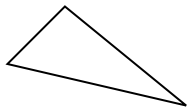
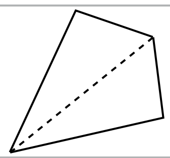
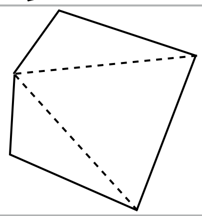
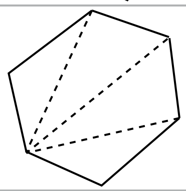
The next two items deal with angles in polygons. The first item (M02_07) is about the relation between the interior angles and the number of sides of a polygon.

M02_07

Interior Angles

Jackson was investigating the properties of polygons. Jackson made up the table below to see if he could find a connection between sides and angles.

A. Complete the table by filling in the blank spaces.

Polygon	Number of Sides	Number of Triangles	Sum of Interior Angles
	3	1	1 × 180°
	___	___	___ × 180°
	___	___	___ × 180°
	___	___	___ × 180°

B. Put the correct number into the box.

The sum of the interior angles of a polygon with 10 sides = $\boxed{}$ × 180°

M042301_1

Nearly two thirds of the sample (57.9 %) filled in the blanks correctly. Completing the table like this is on a concrete level in contrast to item B, where the sum of the interior angles of a 10-sided polygon is to be given. Less than one third (32.4 %) succeeded in filling in the correct number, namely 8 * 180°.

On a still higher abstraction level, where the sum of the interior angles in an n-sided polygon is to be given, $(n - 2)$ * 180°, the solving frequency decreased to barely one tenth (8.1 %).

M02_07

C. Jackson could see a pattern and was able to write an expression using *n* that is true for any polygon. Complete what he wrote.

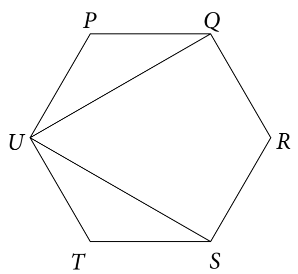
Sum of the interior angles of a polygon with *n* sides = _____ × 180°

M042301_2

A conclusion possible to draw is that the abstraction level had a crucial influence on the solving frequency. A limited number of the pupils mastered this kind of abstract thinking by means of variables. The item should be classified to the domain of algebra and not to geometry.

The second item (M05_09) deals with the angles of a regular hexagon.

M05_09



PQRSTU is a regular hexagon. What is the measure of the angle *QUS*?

Ⓐ 30°
Ⓑ 60°
Ⓒ 90°
Ⓓ 120°

M032205

In order to calculate the angle QUS it is necessary to focus on the two small isosceles triangles QPU and UTS. Symmetrically these two triangles are congruent, since their two respective sides and their intermediate angles are of equal size. A regular hexagon can be divided into four triangles out of which three are of equal size. The sum of all the angles is therefore four times 180°, which is 720°. Since there are six interior angles of the same size in a regular hexagon, one of them is 120°. Therefore the top angle in each of the small congruent triangles is 120°. Since the sum of the angles in a triangle is 180°, the other two are 30° each. The angle PUT, which is 120°, consists of the two angles of 30° each and of the unknown angle QUS. Taking 30° + 30° from 120° makes the requested angle QUS 60°.

The correct alternative, 60°, was picked out by about half of the pupils (55.2 %). This is a result that could also be arrived at by just looking at the figure.

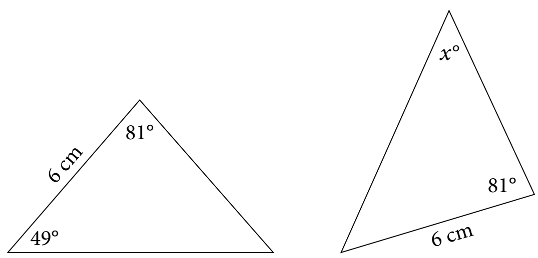
One sixth of the pupils (13.9 %) chose the most attractive distracter 120°. The multiple choice design facilitated the item considerably, since it is possible to visually estimate the size of the angle.

Another possible strategy for solving the item is to draw the diagonal QS. Then three small isosceles triangles, which are congruent, are obtained. Hereby the sides of the triangle QUS are of equal size and makes the triangle equilateral. In an equilateral triangle, the angles are of equal size, 60°. Depending on the chosen solving strategy, the item will be of various degree of difficulty.

9.2.4 The Concept of Triangle

The two consecutive items both deal with triangles. In the first item (M01_08), there are two congruent triangles. The measures of some of the sides and angles are given. They seem, however, chosen in a way that makes it impossible to figure out the correct alternative by just looking at the triangles. The right triangle is a reflection of the left, since the sides of 6 cm is on different sides of the angles of 81°. This makes the item extra difficult.

M01_08



The triangles shown are congruent. The measures of some of the sides and angles are given. What is the value of x?

(A) 49
(B) 50
(C) 60
(D) 70
(E) 81

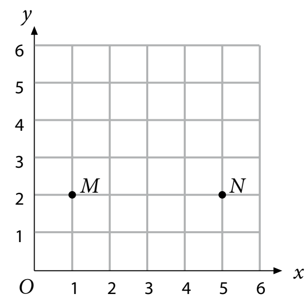
M022062

Getting the alternative of 49°, the triangles have not been regarded as each other's reflection (39.9 %). The alternative of 50° is the correct and was chosen by nearly half of the sample (42.1 %) while the other distracters were low-frequent.

The main difficulty in this item was the ability of seeing the two triangles as each other's reflection. It is a necessary requirement to be capable to discern that the unknown angle faces the 6 cm side.

In the second item (M07_10), the familiarity with the concept of isosceles triangles is tested.

M07_10



Two points *M* and *N* are shown in the figure above. John is looking for a point *P* such that *MNP* is an isosceles triangle. Which of these points could be point *P* ?

Ⓐ (3,5)
Ⓑ (3,2)
Ⓒ (1,5)
Ⓓ (5,1)

M032294

Less than half of the pupils (48.0 %) picked out the correct multiple-choice alternative (3, 5). The coordinates of the second alternative (3, 2) represents a point, placed on a line through the two points M and N, a circumstance that makes it impossible to construct a triangle. Yet, a group of pupils (16.9 %) chose this alternative. The third alternative (1, 5) was low-frequent, while the fourth (5, 1) was chosen by less than one fourth of the pupils (24.1 %). The coordinates (5, 1) represents a point that makes it impossible to construct an isosceles triangle. The non-alternative coordinates (1, 6) would, however, have been suitable for constructing an isosceles triangle.

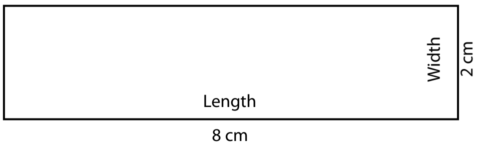
To discriminate the order of the *x*- and *y*-coordinates and to determine the position of the point with the help of its coordinates was found to be the major difficulty.

9.2.5 The Concept of Rectangle

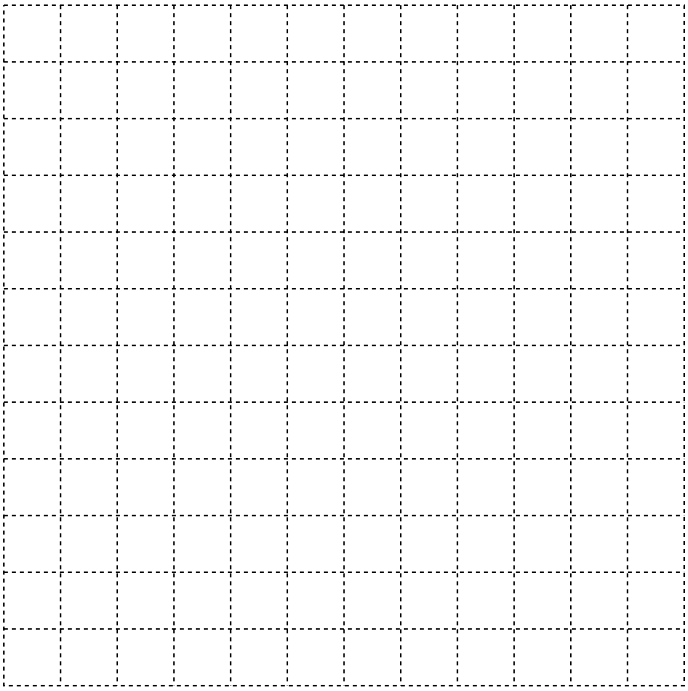
Not only the concept of rectangle but also those of increase and decrease were tested in item M01_11. According to the information given in A, the pupils are to draw a rectangle on a grid. The length of the original rectangle is to be decreased while its width is to be increased.

Changing the length to three fourths of 8 cm and the width to two and one half of 2 cm caused the pupils difficulties. Only a little more than one fifth (22.0 %) solved the item completely while about an equal proportion (22.1 %) solved it partly.

M01_11



A. On the grid below, draw a rectangle whose length is three-fourths the length of the rectangle above, and whose width is two and one-half times the width of the rectangle above. Label the length and width of the new rectangle in centimeters on the figure. Each square on the grid is 1 cm by 1 cm.



B. What is the ratio of the area of the original rectangle to the area of the new one?

M022234

In part B, the multiplicative relationship between the areas of the original and the new rectangle is to be calculated. This caused several pupils great difficulties and only a few (1.4 %) managed to solve the item completely.

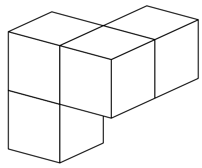
The very calculation of the areas could not have caused any particular problems, since the formula for the area of the rectangle is directly applicable. Instead, the difficulty consisted in understanding the multiplicative relationship between the two areas and in deciding which to come first in this comparison. The correct answer is 16 to 30. According to the assessment guidelines, the relation of 30 to 16 was not considered a completely correct solution. Evidence from previous research confirms that special difficulties arise when pupils shall carry out multiplicative comparisons as in B of this item (Greer, 1992).

9.2.6 Two-Dimensional Representations of Three-Dimensional Objects

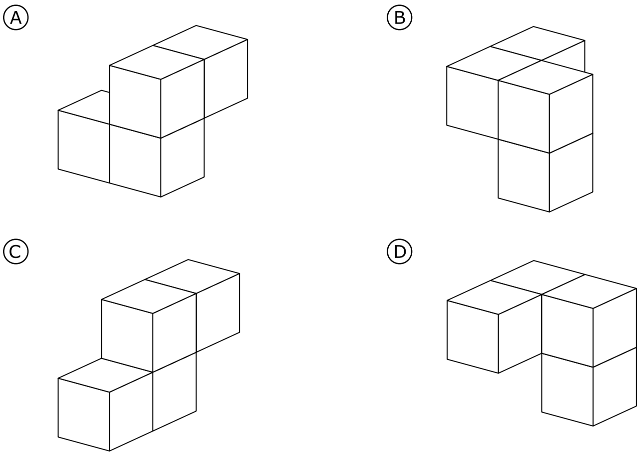
The relation between two-dimensional representations of three-dimensional objects is treated in this section. In item M01_03, a two-dimensional representation of a three-dimensional object is to be rotated.

M01_03

This object will be turned to a different position



Which of these could be the object after being turned?



M022049

The four alternatives are two-dimensional representations of three-dimensional objects, one of which represents the rotated object. The correct alternative D was also most frequent (65.6 %). The most frequent distracter, B, represents the mirrored representation of the rotated object (15.6 %).

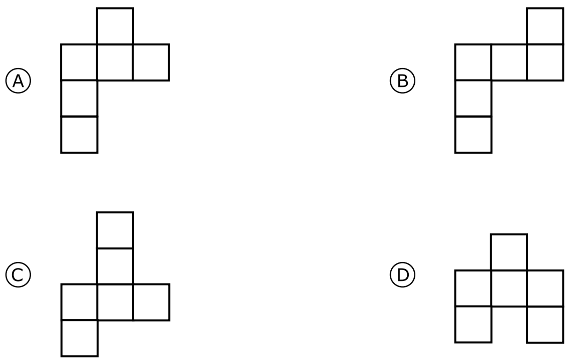
Being successful and solving the problem is apparently connected with the pupils' spatial ability. Experiences of two-dimensional representations of three-dimensional objects supposedly play a crucial role (Ryu, Chong & Song, 2007).

Out of four two-dimensional figures, in item M02_09, one is possible to fold into a cube.

The two alternatives C and D were almost equally frequent (39.8 %; 40.2 %) of which the least frequent, C, was the correct. The alternatives A and B were both low-frequent. Yet again, spatial experiences are of crucial importance in order to see which of the four two-dimensional figures is possible to fold into a cube, which is a three-dimensional object (Ryu, Chong & Song, 2007).

M02_09

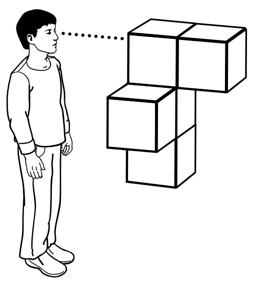
Which net when folded forms a cube?



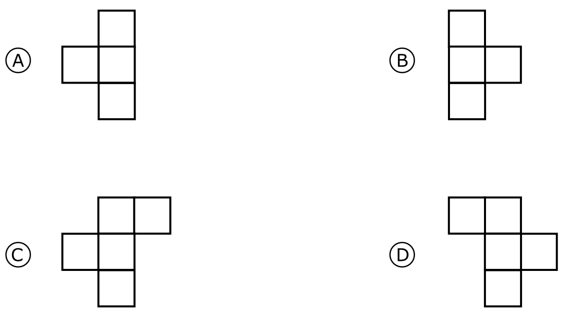
M042265

The third item (M04_09) differs from the previous one, since it tests the ability of putting oneself into another person's shoes. The person stands in front of a formation of five cubes and sees one of the four two-dimensional shapes given.

M04_09



The solid is made of 5 small cubes.
Which shape does the person in the diagram see?

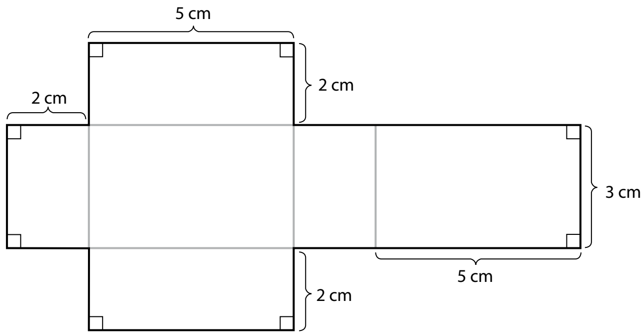


M042279

The pupils are to figure out which of the four two-dimensional projections the person sees. A clear majority of the sample (75.9 %) solved the problem, which also tested the ability of taking another person's perspective.

In the fourth item (M05_04), a shape is to be folded up so that a rectangular box is created and whose volume is to be calculated.

M05_04



When the shape shown above is folded up, it will make a rectangular box.
What is the volume of the box?

Answer: _______________ cm^3

M032344

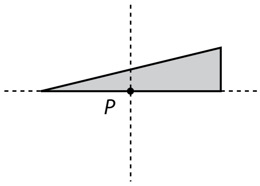
More than one fifth of the pupils (21.3 %) succeeded in solving the item. The pupils of the EU/OECD-countries were on average more successful, however, (39.6 %). The results of this item further emphasize the role of spatial experiences in the mathematics teaching (Ryu, Chong & Song, 2007).

9.2.7 Conformed Depictions of Geometrical Figures

Conformed depictions by rotations around a point are not part of the attainment level of the syllabus for mathematics grade nine. Rotation is, however, a very central part of the model of the concept of angle that is used in school mathematics. In item M03_04, a triangle is rotated half a turn around a point.

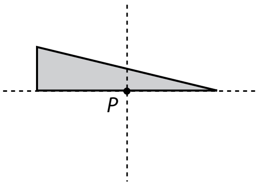
M03_04

A half-turn about point *P* in the plane is applied to the shaded figure.

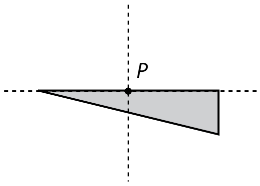


Which of the following shows the results of the half-turn?

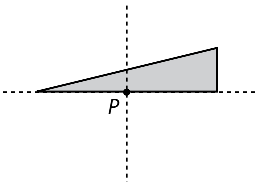
Ⓐ



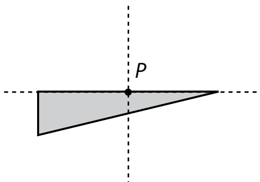
Ⓑ



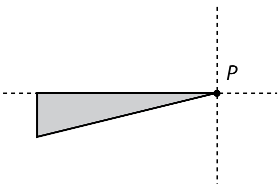
Ⓒ



Ⓓ



Ⓔ



M022105

Five multiple choice alternatives are given out of which the correct, D, was highly frequent (54.2 %). The two frequent distracters, A and B, both represent the mirrored images of the triangle; A in a vertical and B in a horizontal line (12.6 %; 17.8 %). The two low-frequent alternatives were C and E. The alternative C, (5.0 %), which is the same figure in the same position as the original, is thus not rotated. The alternative E, (6.3 %) is, however, rotated but also translated that is displaced to the negative side of the origin of coordinates.

Without the harmonization with the syllabus, the solving frequency was relatively high.

9.2.8 Summary and Conclusions

As reported in the international research review, there is support that pupils mix up the two concepts of perimeter and area. This showed to be applicable to Swedish pupils as well.

It could be suspected that geometry has been more procedurally than conceptually taught. When the formula for calculating the area of a triangle could be applied directly, the solving frequency was relatively high. However, when a conceptual adaptation was required, the solving frequency decreased significantly. Familiarity with the concept of area of a circle is not only the knowledge to calculate the area of a circle but the understanding of how the area units can be made to cover the circle surface. It is for example conceptually important to see how many times larger the area of a circle with the radius r is compared to a square with the side r.

Turning to the additive character of the general concept of area, a majority of the pupils seemed to be familiar with it. A too large proportion of the pupils, however, did not seem to be acquainted with this character of the concept of area.

Conceptions of the conceptual models of the angle concept are known to be problematic. There are three competitive conceptual models. Despite their disadvantages, apparently a conceptual image as support is needed. The pupils showed to have acquired the knowledge of the sum of the interior angles of a triangle, 180°. This knowledge was applicable in most of the items. Also necessary knowledge is that half a turn is 180°.

Angles are classified regarding position and size. Concerning position, there are vertical, alternate, interior and exterior angles. Another characteristic of position is that an angle can be faced to a side. In an isosceles triangle, there are one top angle and two base angles, which are of equal size. Concerning size, there are right, obtuse and acute angles. The terminology of position and size needs to be frequently taught because unawareness of these concepts is the probable reason for the moderate solving frequencies.

Also the level of abstraction in the test items seemed to have a crucial influence on the solving frequency. Concurrent increase of abstraction resulted in decrease of the solving frequency.

Moreover, the role of spatial experience in mathematics teaching seems to play a crucial role for the understanding and interpretation of two-dimensional representations of three-dimensional objects.

Despite the lack of matching with the syllabus, there were pupils who solved the test items on spatial ability.

Pupils' Exposed Calculation Strategies in the National Assessment Test

10. Pupils' Exposed Calculation Strategies in the National Assessment Test

In order to further strengthen the validity of the TIMSS-results, an analysis of 507 pupils' solutions of specially selected items in the national assessment test of 2007 was performed. Since subtractions in the TIMSS-project showed to be especially tricky for the grade 4 pupils, solutions of subtractions with and without trading from the national assessment test for grade 5 pupils were analysed. Also text problems that were modelled by subtractions were examined. The design of certain items in the assessment test was particularly interesting because a text problem modelled by the subtraction 15 – 6 was presented on one page, while the corresponding non-textual subtraction 15 – 6 on another page. The pupils were requested to answer a number of pre-printed questions and by doing so revealed their presumptive solving strategies. The fact that the same numbers emerged in two different problem contexts made it possible to compare the pupils' calculation strategies and solutions.

The selected test items of the national assessment test represent three different aspects of calculations. One concerns the understanding of the number concept and subtraction, while another subtraction without trading. The third aspect of calculation is subtraction with trading.

10.1 Conceptions of Numbers and Subtraction

The first item, 1 000 – 2, had a high solving frequency (92.0 %). None of the items of the TIMSS-project had such a high solving frequency.

Table 13 Categories Representing Pupils' Different Strategies when Calculating 1000 – 2, Grade 5, n = 501

Categories	Frequency	Relative Frequency (%)	Typical Answer
Number facts	461	92.0	998
Standard algorithm	5	1.0	998
Jumping Strategy	7	1.4	998
Concatenated Structure	28	5.6	9998. 8
Total	**501**	**100.0**	

Most of the pupils' mistakes were classified as concatenated in their structure (Fuson, 1992). There were pupils who were uncertain of the numbers that come just before 1 000. The most frequent answer was 9998. (See table 13).

The pupils should have solved the item, 15 – 6, by number facts which also a majority of the pupils (82.7 %) did as is to be seen in Table 14. Relatively few pupils (3.6 %) failed in their calculations. Besides number facts, the most frequent calculation approach was the jumping strategy, which was used by a minor group of pupils (10.9 %), who also stated that finger counting was their regular way of counting.

Table 14 Categories Representing Pupils' Different Strategies when Calculating 15 – 6, Grade 5, n = 504

Categories	Frequency		Relative Frequency (%)		Typical Answer	
	Corr	Incorr	Corr	Incorr	Corr	Incorr
Number facts	417	6	82.7	1.2	9	-
Standard algorithm	7	-	1.4	-	9	
Compensating Strategy	6	2	1.2	0.4	9	-
Splitting Strategy, no Trading	-	9	-	1.8	-	11
Splitting Strategy, Trading	1	-	0.2	-	9	-
Jumping Strategy	55	1	10.9	0.2	9	-
Total	**486**	**18**	**96.4**	**3.6**		

The relatively high solving frequency (96.4 %), at first sight, seems satisfactory. Looking closer and reconsidering it, however, 18 pupils of this sample of 500 corresponds to 3 600 pupils of the whole population of 100 000 pupils. By extrapolation, this gives about 180 classes á 20 pupils who are not capable of solving an elementary subtraction, like 15 – 6. Bearing this in mind, relatively high solving frequencies are not sufficient but must be very high for being acceptable.

Table 15 illustrates the pupils' strategies when calculating 17 – 8. Besides the usual strategies, the category of deviant results is added and shows the solving frequency of the non-textual problem, 17 – 8, and of its corresponding text problem. If classified as deviant results and correct, it has, however, been incorrect when solving the same item in the corresponding text problem. If, on the other hand, classified as deviant results and incorrect, the subtraction has been correct in the text problem. Adding these two frequencies, it will represent the number of pupils, who have exposed not only correct but also incorrect calculation strategies for one and the same item. In table 15, there is for example one pupil, who exposed an incorrect result of the non-textual problem and a correct result of the subtraction when in a text problem.

Table 15 Categories Representing Pupils' Different Strategies when Calculating 17 – 8, Grade 5, n = 502

Categories	Frequency		Relative Frequency (%)		Typical Answer	
	Corr	Incorr	Corr	Incorr	Corr	Incorr
Number facts	443	13	88.2	2.6	9	-
Standard algorithm	6	-	1.2	-	9	-
Splitting Strategy, no Trading	-	3	-	0.6	-	11
Splitting Strategy, Trading	2	-	0.4	-	9	-
Jumping Strategy	26	3	5.2	0.6	9	-
Transformation Strategy	4	1	0.8	0.2	9	-
Deviant Results	-	1		0.2	-	-
Total	**481**	**21**	**95.8**	**4.2**		

The main proportion of the pupils in the sample solved the item by means of number facts. Those who used the jumping strategy also took advantage of their fingers when counting.

Considering the circumstances reported above, this relatively low frequency of pupils who failed on the item corresponds, however, to about 4200 pupils, which is an unacceptable number of pupils.

A similar TIMSS-item (M04_01) that tested the understanding of the number concept had an equally low solving frequency, which is a result that further strengthens the validity of the two studies.

10.2 Subtraction without Trading

This section deals with two items on subtraction that does not require trading. The two-digit subtraction item had, as expected, a somewhat higher solving frequency than the three-digit item.

The pupils' exposed calculation strategies as reflected in their solutions are displayed in table 16. The main part of the pupils (89.2 %) succeeded in solving the item of 57 – 34. Out of these one fourth (24.1 %) solved it by means of the standard algorithm, which thus contributed to the high solving frequency. Without the algorithmic calculations, the picture would probably have seemed more negative. The two strategies jumping and splitting have become more frequent compared to the previous test items. The deviant results category shows that two pupils have two strategies each for one and the same item, one correctly and one incorrectly applied.

Table 16 Categories Representing Pupils' Different Strategies when Calculating 57 – 34, Grade 5, n = 503

	Frequency		Relative Frequency (%)		Typical Answer	
Categories	**Corr**	**Incorr**	**Corr**	**Incorr**	**Corr**	**Incorr**
Number facts	248	36	49.3	7.2	23	-
Standard algorithm	121	1	24.1	0.2	23	-
Splitting Strategy, no Trading	24	5	4.7	1.0	23	-
Splitting Strategy, Trading	-	6	-	1.2	-	17
Jumping Strategy	55	4	10.9	0.8	23	-
Mixed Strategy	1	-	0.2	-	-	-
Deviant Results	1	1	0.2	0.2	-	-
Total	**450**	**53**	**89.2**	**10.8**		

The following item 257 – 123 of two three-digit numbers does not require any trading. In table 17, the solving frequencies are presented and indicate that the mistakes have increased and concern in particular number facts (6.3 %), the jumping strategy (2.8 %) and the splitting strategy for trading (1.8 %). This last strategy was applied to a subtraction that does not require trading and ended in the typical answer of 126.

Table 17 Categories Representing Pupils' Different Strategies when Calculating 257 – 123, Grade 5, n = 494

	Frequency		Relative Frequency (%)		Typical Answer	
Categories	**Corr**	**Incorr**	**Corr**	**Incorr**	**Corr**	**Incorr**
Number facts	187	31	37.9	6.3	134	-
Standard algorithm	175	7	35.4	1.4	134	-
Splitting Strategy, no Trading	26	3	5.3	0.6	134	-
Splitting Strategy, Trading	-	9	-	1.8	-	126
Jumping Strategy	33	14	6.6	2.8	134	-
Transformation Strategy	1	7	0.2	1.4	134	140
Deviant Results	-	1	-	0.2	-	-
Total	**422**	**72**	**85.5**	**14.5**		

The corresponding test items in the TIMSS-project all required trading. Consequently a comparison is not relevant (See section 8.1.3).

One third (35.4 %) of the pupils made the calculation of this item by means of the standard algorithm and hereby contributed markedly to the total solving frequency of 85.5 %. Without this third, the frequency would probably have decreased noticeably.

10.3 Subtraction with Trading

The three next test items are about subtraction with trading. Decisively lower solving frequencies turned up, except for the last item, which involved the rather close numbers of 203 and 198, which probably made the item easier. In table 18, the solving frequencies of the different calculation strategies applied to the subtraction 91 – 59 are displayed. It is especially obvious in items of this kind that the splitting strategy without trading is applied incorrectly and gives the typical answer of 48. This was relatively high-frequent (11.4 %). Also the transformation strategy for additions was incorrectly applied to the subtraction (4.5 %) and gave the result of 30 that is close to the correct one, 32.

Table 18 Categories Representing Pupils' Different Strategies when Calculating 91 – 59, Grade 5, n = 507

	Frequency		Relative Frequency (%)		Typical Answer	
Categories	**Corr**	**Incorr**	**Corr**	**Incorr**	**Corr**	**Incorr**
Number facts	77	6	15.2	1.2	32	-
Standard algorithm	160	12	31.6	2.4	32	-
Splitting Strategy, no Trading	-	58	-	11.4	-	48
Splitting Strategy, Trading	31	2	6.1	0.4	32	-
Jumping Strategy	42	4	8.3	0.8	32	-
Mixed Strategy	2	4	0.4	0.8	32	-
Transformation Strategy	2	23	0.4	4.5	32	30
Deviant Results	58	26	11.4	5.1	-	-
Total	**372**	**135**	**73.4**	**26.6**		

The most interesting about table 18 is the category of deviant results. To the non-textual item, 91 – 59, 58 pupils gave a correct answer but to the corresponding text item an incorrect, however. The inverse also occurred, namely a wrong answer on the non-textual item and a correct on the corresponding text item. This was done by 26 pupils. Taken together, 84 pupils showed to master not only one calculation strategy that was correctly applied but also one that was incorrectly applied. The encoding of the problem was rather uncomplicated because it consists in a change-take-from situation. Minor influence of the encoding process on the result cannot, however, be excluded and could explain the asymmetric results. Extrapolating from the sample to the population, 84 pupils correspond to 17000 pupils. Unknown is, however, the size of the proportion of the pupils who have parallel calculation strategies that were correctly or incorrectly applied.

This item (91 – 59) can be compared to one of the items of TIMSS. There the difference of the weight between a boy and his cat is to be calculated, 62 – 57 (M01_08 or item number M031301). These two subtractions of two-digit numbers both require trading. Due to the relative closeness of the two numbers of the subtraction 62 – 57, this is, however, easier to perform than 91 – 59. But the encoding of the 'cat problem' could have caused difficulties since it is a comparison situation. Nevertheless, it is possible to draw the conclusion that the validity of the two studies is strengthened by the similarity of their solving frequencies 72.0 % and 73.4 %.

Table 19 illustrates the solving strategies that have been applied to the subtraction 151 – 126.

Table 19 Categories Representing Pupils' Different Strategies when Calculating 151 – 126, Grade 5, n = 503

	Frequency		Relative Frequency (%)		Typical Answer	
Categories	**Corr**	**Incorr**	**Corr**	**Incorr**	**Corr**	**Incorr**
Number facts	94	12	18.7	2.4	25	-
Standard algorithm	167	20	33.2	4.0	25	-
Splitting Strategy, no Trading	-	46	-	9.1	-	35
Splitting Strategy, Trading	28	-	5.6	-	25	-
Jumping Strategy	54	6	10.7	1.2	25	-
Mixed Strategy	4	-	0.8	-	25	-
Transformation Strategy	3	8	0.6	1.6	25	-
Deviant Results	39	22	7.8	4.4	-	-
Total	**389**	**114**	**77.3**	**22.7**		

The solving frequency (77.3 %) is approximately the same as in the previous item (73.4 %). Relatively high-frequent (9.1%) is the incorrectly applied splitting strategy. Also in this item, a certain number of pupils exposed both correct and incorrect calculation strategies (12.2 %). So, one and the same pupil can in a certain context expose a correctly applied calculation strategy and in another an incorrectly.

Considering the results from the last two items (91 – 59, 151 – 126) approximately one third of the pupils showed to be familiar with the standard algorithm for subtraction. This fact casts light on the relatively low solving frequency (20.4 %) of the TIMSS-item M01_02 or item number M031106 that tests pupils' knowledge of the standard algorithm for subtraction. It seems as if a number of pupils have no experience of the standard algorithm but only of the various mental calculation strategies.

In table 20, the solving frequency of the subtraction 203 – 198 (87.6%) has a different character. Since the two terms of the subtraction are relatively close, the calculation is facilitated.

The without-trading splitting strategy was incorrectly applied to a subtraction that requires trading and led to the incorrect answer of 195. On the grounds that the two numbers are so close, it would be expected of the pupils to react reasonably and come to the conclusion that the answer of 195 is incongruous and cannot be correct. This situation can be enlightened by the extended variation theory, namely that several conceptual attributes simultaneously need to be comprehended for reaching full understanding of the number concept. Thus, both the calculation strategy and the numerosity of the numbers, the internal number line, simultaneously have to be available to the pupil. If these conceptual attributes are separately experienced, there will be no reaction to the incongruity of an answer however deviant it may be.

Table 20 Categories Representing Pupils' Different Strategies when Calculating 203 – 198, Grade 5, n = 501

Categories	Frequency		Relative Frequency (%)		Typical Answer	
	Corr	Incorr	Corr	Incorr	Corr	Incorr
Number facts	261	9	52.1	1.8	5	-
Standard algorithm	93	10	18.6	2.0	5	-
Splitting Strategy, no Trading	-	25	-	5.0	-	195
Splitting Strategy, Trading	13	2	2.6	0.4	5	-
Jumping Strategy	44	2	8.8	0.4	5	-
Mixed Strategy	2	-	0.4	-	5	-
Transformation Strategy	-	4	-	0.8	-	6
Deviant Results	26	10	5.2	2.0	5	-
Total	**439**	**62**	**87.6**	**12.4**		

Also the solving of one and the same item, a non-textual and its corresponding textual item, showed the pupils' (7.2 %) more or less simultaneous exposure of at least one correctly and one incorrectly applied calculation strategy.

In brief, the solving frequencies of the national assessment test support the validity of the TIMSS-project. The fact that one and the same pupil simultaneously can have both incorrect and correct calculation strategies is clearly displayed in the pupils' solutions. Comparable results were arrived at in the interviews of the pupils in the Lilla Edet-project. From this can be concluded that having learnt to apply a calculation strategy correctly does not guarantee that this application will always be exposed. Instead, incorrect applications may well be used.

Discussion

11. Discussion

First, a survey of the conclusions of the main results of the present study will be presented. Then the results will be related to previous research. An argumentation whether the aim of the present study has been reached will then be given. After that different aspects of the limitation of the present study will be analysed and accounted for. Finally, taking the present results as a point of departure, suggestions of future research will be put forward.

11.1 A New Insight into the Character of the Mistakes

Altogether, there is evidence of the results that pupils principally do not make fortuitous calculation mistakes. Instead, the mistakes are significantly more profoundly thought-out and rest upon insufficiently developed understanding of relevant and related concepts and conceptual models. The conceptual models used in teaching often have a narrow range of application. Outside the range the conceptual models convey little or no guidance for the pupils' thinking. Moreover, there are conceptual models of only limited operational guidance, especially outside their respective range of application. One example is the Object Model that is used within algebra in order to facilitate simplifications of additive expressions, like $2a + 3b + 3a$. According to the Object Model, the letters a and b represent apples and bananas for instance. The structure that the model is supposed to convey is that the different kinds of fruit are to be added separately. When applied outside additive expressions, however, the result becomes incorrect and the pupils experience no guidance of the model in their solving of the problem. This was shown in the item of a multiplicative expression to be simplified, $2a^2 * 3a$. The operation of multiplying apples by apples lack conceptual meaning.

The results of this report also show that a pupil has not only one but several parallel ways of understanding one and the same concept. Such different conceptions are often contextually dependent. Even though contradictory and inconsistent, they yet exist side by side. A pupil can for example expose ways of understanding the concept of variable as a specific unknown number and as a concrete symbolic representation.

Furthermore, it was shown that the pupils' mistakes had their roots in the applications of the calculation strategies in inaccurate contexts rather than in the misunderstanding of the strategy as such. The two different versions of the splitting strategy, one for subtraction without trading and one with trading, serve as an example. Frequently the pupils incorrectly applied the splitting strategy for subtraction without trading to subtraction that requires trading.

There was also convincing evidence that a number of different strategies were applied to a calculation by one and the same pupil, some of which were correct and some of which were incorrect. This was done to one and the same calculation but in two different items, one in a text problem and one in a non-textual. So, one and the same pupil exposed a correct calculation of a two-number item in one context and an incorrect calculation of the same numbers in another context.

The main purpose of the procedural approach to mathematics teaching is that the pupils shall learn to master various calculation procedures. Not being able to perform the procedures correctly, the pupils are considered not to have mastered the calculation procedures. This dichotomy, a correct contra an incorrect calculation performance, led to the construction of diagnostic test items. These showed whether or not a pupil mastered a unit that had been taught in class. Some of the diagnostic test items were based on a hierarchical structure of prior-knowledge and implied that lack of prior-knowledge caused the pupils' mistakes. However, the main results of the present study show that prior knowledge alone does not give a sufficient picture of the pupils' obstacles in their mathematical development. Applying a calculation procedure correctly but in an incorrect context or using a conceptual model outside its application range does not primarily imply lack of traditional prior-knowledge. Development of mathematical knowledge is a much more complex mental activity. Therefore, also other causes than only prior-knowledge have to be taken into account.

11.2 Pupils' Knowledge in Relation to Relevant Research Results

In the following, the picture of pupils' mathematical knowledge will be related to relevant research results. First, grade-four pupils' understanding of the number concept and their arithmetic skill and secondly, their knowledge of geometry will be accounted for. Thirdly, grade-eight pupils' knowledge of algebra will be related to relevant research as will their knowledge of geometry. Lastly, grade-five pupils' knowledge of subtraction as it appears in the national assessment test will be analysed with respect to previous research.

11.2.1 Arithmetic Knowledge and Understanding of the Number Concept, Grade 4

The concept of place value was tested in one item, where the ones, the tens and the hundreds were given. Reversing of the numbers occurred. This is a phenomenon that is known from national research (Johansson, 2005; Bentley, 2008b).

Regarding subtraction, various mistakes were exposed. Most of them were related to the encoding of the text problem. The principally different problem situations modelled by subtractions are change-take-from, equalise and compare. Fischbein et al., (1985) claimed that it is the change-take-from situation that pupils learn and take advantage of. The problem situations represented by equalising and comparing, on the whole, do not seem to be enlightened or exemplified in the teaching. According to Fuson (1992), comparison situations are difficult to encode. This also showed to be the case in the test items, where the text problems were to be encoded as comparison situations. Consequently, the solving frequencies on these items were low. Even though one of the items represented a change-take-from situation, the solving frequency was relatively low. The explanation for this was not to be found in the encoding process, however, but in the number interval of the item.

Another recurrent mistake concerned the carrying out of the standard algorithm for subtraction. Only one fifth of the pupils succeeded in completing the algorithm. The national assessment test for grade five showed that only about one third of the pupils mastered the standard algorithm. Transferring this result to the TIMSS-test strengthens the recurrence of the mistakes, since also in the same kinds of items in TIMSS the solving frequency was low.

Also known from international research is the mistake of the application of the splitting strategy to incorrect contexts. These kinds of mistakes were, however, relatively high frequent, a fact that is confirmed by the TIMSS-results. Concerning certain items, more than half of the pupils exposed the mistake of applying the calculation procedure for subtraction without trading to subtraction that requires trading.

In multiplication, however, the encoding process showed to be the stumbling block. Items of problem situations to which simple intuitive multiplication models of the type repeated addition could not be applied had low solving frequency. When, on the other hand, an intuitive model functioned and solved the problem, the solving frequency enhanced. This outcome is known from previous research (Fischbein et al., 1985). One of the items consisted in both a multiplicative and a subtractive comparison situation. Hardly half of the pupils managed to encode that situation correct. As has been reported, both multiplicative and subtractive comparison problem situations are considered to increase the level of difficulty of an item, not least when they occur simultaneously in one and the same item (Fuson, 1992; Greer, 1992). Also, this part of the results is in harmony with previous research of the field.

In division, the encoding of the text problems was the crucial difficulty. When the problem situation was represented by a quotitive division, the solving frequency decreased as expected, since pupils acquire quotitive division relatively late in their schooling according to Fischbein et al. (1985). When short division was not possible to apply directly to the numbers of an item, the solving frequency was negatively affected. Accordingly, Swedish pupils' difficulties concerning division problems are consistent with the results from international research (Greer, 1992).

Vulgar fraction is also a field where the pupils did not succeed very well. A well-described mistake about addition of fractions is that the pupils do not make any difference between the nominator and the denominator but separately add the nominators and the denominators. Reasons for this have shown to be not only insufficient teaching of vulgar fraction but also the use of inadequate conceptual models (Davis, 1997; Siegler, 2003).

Proportionality is another field that generally caused the pupils difficulties. Not only were proportion and correspondence mixed up but also proportional increase and additive increase. Research on proportionality has shown that pupils have difficulties in distinguishing the concepts of proportion and correspondence and of the sub-concepts of proportional increase and additive increase (Bentley, 2008a; Hart, 1981).

Taken together, the results of the analyses of the pupils' solutions of the national assessment test and of the TIMSS-test on understanding of the number concept and arithmetic knowledge are consistent with research results in this field.

11.2.2 Geometry, Grade 4

The confusion of the two concepts area and perimeter seems to be a problem for Swedish grade-four pupils. As the measure of the size of a rectangle, they add its length and breadth. In one of the test items, approximately half of the pupils exposed such performance. Especially younger pupils use linear measures, as for example the sum of the length and breadth, for giving the size of a rectangle, (Clements & Stephan, 2003). The high frequency of such an alternative is rather a reflection of a non-problematized teaching of the use of linear measures for two-dimensional characteristics.

Also the conservation of the area concept showed to be problematic and accordingly arrived at low solving frequencies. As has been reported in the research review, pupils are not favoured by a numerical focus in the teaching approach (Hiebert, 1981; Douady & Perrin, 1986; Stigler & Hiebert, 1999; Patronis & Thomaidis, 2008). The descriptions of the two lessons about the area concept in the TIMSS-video study, one in Japan and one in the US, clearly illustrate the difference between a conceptual and a procedural focus (See section 4.3.2). The procedural approach often focuses on the solving of the items by the application of the calculation procedures that have been mastered. When a conceptual adaptation is required, however, the pupils are not able to keep up.

The pupils' classifications of the attributes of triangles were successful.

To examine the number of area units that are needed to cover a given surface is known as an application of the principle of measurement. This principle that is modelled by a quotitive division describes how the measure of the area is arrived at. The solving frequencies of these kinds of items were relatively low, which is in agreement with research on quotitive division (Greer, 1992).

By means of a two-dimensional representation of a three-dimensional object, the pupils' spatial ability was tested. Approximately half of the pupils solved the item. From international research is known that spatial ability is in general developed rather late unless special training with, for example, computer animation is offered. Ryu, Chong and Song (2007) found that even talented pupils had difficulties in developing spatial ability.

The key component in one of the conceptual models of the angle concept is rotation. One third of the pupils had mastered the concept of angle with the help of rotation. That so few pupils understood the concept of angle by means of rotation is in conformity with research on understanding of the angle concept. The very rotation seems to be the major difficulty. Foxman and Ruddock (1983) reported that only 4 % of the 15-year old pupils spontaneously mentioned rotation when requested to describe the angle concept. Items on reflection and symmetry had about twice as high solving frequencies. No special difficulties concerning these phenomena were found in international research.

The main part of the mistakes that the Swedish pupils made in geometry has been described in international research. Therefore, the results of the analyses of the present study are considered to be in accordance with previous research.

11.2.3 Algebra, Grade 8

In the test items on equations, it was shown that the pupils' understanding of the equality sign played a significant role for their possibilities to solve equations with x-terms on both sides of the sign. Only a few pupils seemed to have developed the static understanding of the equality sign, a circumstance that is in conformity with relevant research (Ginsburg, 1977; Behr, Erlwanger & Nichols, 1980; Kieran, 1981; Falkner, Levi & Carpenter, 1999; Carpenter, Franke & Levi, 2003).

Simplifications of expressions were relatively easy for the pupils when the Object Model was possible to imply directly. But when multiplicative expressions were to be simplified, the Object Model was of no use, since it could not be applied to multiplicative expressions. Consequently, the solving frequencies were low. This result harmonizes with previous research (Küchemann, 1981; Wagner, 1983; Booth, 1984; Philipp, 1992; MacGregor & Stacey, 1997; Bentley, 2008a).

In the test items on functions, in which formulas and equations with two unknowns were tested, the pupils' ways of understanding the concept of variable played a decisive role. Since it is the context that determines the meaning of the variable, it is of special importance that the variable is understood in the intended way. In a functional context, the variable shall be understood as a generalised number. The solutions of a number of non-released items showed a variety of ways of misunderstanding the concept of variable, misconceptions that can explain the low solving frequency of the test item on functional relationships. These kinds of misunderstanding the concept of variable are known from research (Küchemann, 1981; Wagner, 1983; Booth, 1984; Philipp, 1992; MacGregor & Stacey, 1997; Bentley, 2008a).

In one of the items, there was no variable letter denotation, which made the level of abstraction low. This resulted in a relatively high solving frequency.

The remaining items of which one concerns inequalities are not covered by the attainment targets of the syllabus for this age group.

11.2.4 Geometry, Grade 8

The two concepts, perimeter and area, were mixed up by the grade-eight pupils. There is research evidence of the inability to distinguish the perimeter from the area of a square (Clements & Stephan, 2003).

In the Western World, the teaching of the concept of area normally has a procedural direction (Stigler & Hiebert, 1999). Hereby, the area calculation formula is directly applied to the numbers given in the problem. If, however, a conceptual adaptation is necessary in order to get the numbers for the calculation of the area, the formula is not possible to apply straight forward. In one of the items, the height of a triangle was not explicitly given. For this reason, the solving frequency was relatively low. In another item, besides the possibility to use a procedural application, a conceptual application was possible, which was not discovered by all pupils, however. In the conceptual application, it was possible to make use of a grid for determining the size of the area by counting its square units. Since this possibility was not discovered by the majority of the pupils, the solving frequency was low.

The additive character of the area concept was only mastered by an excessively small group of pupils. Only half of the pupils took advantage of dividing a given shape into shapes, whose areas were possible to calculate and thus reach the area of the original shape. Experiences of the additive character of the area concept are crucial for pupils' understanding of the area concept. These are the grounds for the ability of calculating the area of a shape, whose area is not directly possible to calculate, by dividing the shape into other shapes, whose areas can be operationally calculated (Chick & Baker, 2005).

In the items, the concept of angle was frequently tested. Its conceptual model is decisive for the understanding of the angle concept and therefore plays an important role in the solving of the items (Mitchelmore & White, 2000). The ability to classify the position and size of different types of angles was part of several test items. It can be concluded that a decisive prerequisite for comprehending this classification is that the model of the angle concept is correctly understood. The mathematical theory that the sum of the interior angles of a triangle is 180° was most frequently used in the items. Most pupils seemed to master the theory.

Pupils' spatial ability was tested by means of the interpretation of a two-dimensional representation of a three-dimensional object. Experiences of such representations play a decisive role for the pupils' possibilities to solve these kinds of items (Ryu, Chong & Song, 2007). The relatively low solving frequencies confirm the research results of this field.

Rotation is a crucial part of the model of the angle concept, which is frequently used in school mathematics. Despite exposure to rotation, only a relatively small group of pupils solved the corresponding item. The difficulty in acquiring one of the key components of the angle concept is in conformity with previous research, which shows that the concept of angle is acquired relatively late in schooling (Foxman & Ruddock, 1983; Mitchelmore & White, 1998).

11.2.5 Subtraction in the National Assessment Test 2007, Grade 5

The pupils' solutions to the items on subtraction in the national assessment test for grade five can be divided into three types. The first is about the understanding of the number concept. This was tested by means of simple subtraction items and was solved at relatively high frequencies. The corresponding item in the TIMSS-project was solved at approximately the same frequencies, a fact that secured its validity. Concerning the small number interval, the pupils had developed number facts. This was also the case in the Lilla Edet-project (Bentley, 2008b).

The second type is subtraction without trading, which was tested in two items. They had a high solving frequency, a result that is in conformity with the Lilla Edet-project. In the TIMSS-test, there were no subtraction without-trading items.

The third type is subtraction that requires trading. Approximately one fourth of the pupils did not succeed in solving the items on subtraction that requires trading. This can seem as a small proportion, but if extrapolated to the corresponding population of pupils, one fourth corresponds to about 25000 pupils. In both the national assessment test and the TIMSS-test, there are two principally similar subtraction items that require trading. Since the solving frequencies of the two items were almost alike, 72.0 % and 73.4 % respectively, this fact supports the validity of both the national assessment test and the TIMSS-test.

From the analyses of the collected solutions of the national assessment test, it was found that an incorrect application of the splitting strategy was a frequent mistake. This result coincides with what Foxman and Beiszhusern (2002) reported from their research.

The fact that one and the same pupil exposed both correct and incorrect calculation strategies for one and the same item but in different contexts only seems to be known from the study in Lilla Edet (Bentley, 2008b).

11.3 Has the Aim Been Reached?

The first part of the aim was to describe the pupils' solving strategies not only of the released items of the TIMSS-project for grade 4 and 8 but also of the collected pupils' solutions of the national assessment test for grade 5. The pupils' solving strategies of the two projects have been accounted for in the different sections of the results. Further, the pupils' solving strategies have been thoroughly analysed in order to reach the second part of the aim, namely to cast light on the kinds of understanding of a number of key mathematical concepts and on the applications of the calculation procedures that are reflected in the pupils' exposed solving strategies. Thus, by analysing the solving strategies and by taking advantage of relevant research results, it was possible to give a description of the pupils' mathematical knowledge. This gives a multifaceted description of the four graders' knowledge of arithmetic, understanding of the number concept and of geometry. The analyses also give a detailed description of the eight graders' knowledge of algebra and geometry and of the collected solutions of the national assessment test for grade 5. These descriptions have, as has been reported in the first section, showed to be partly different compared to relevant research.

11.4 The Limitations of the Present Study

With reference to the reliability of the present study, the TIMSS-test is performed under strictly controlled conditions. As accounted for earlier, there are three different types of test items, first, multiple-choice items, second, items in which only answers are required and third, items in which full solutions are required.

Only exceptionally, the multiple-choice items caused difficulties in the assessment. This happened when more than one alternative had been chosen. According to the guidelines, a positive assessment should be applied. If two alternatives were chosen, including the correct one, the correct would be accepted on the condition that the second alternative did not expose an understanding contradictory to the correct alternative. At a rough estimate, this only concerned about ten pupils' solutions.

Items where only answers were required were mainly assessed from the point of departure, correct or incorrect. From the answers given in some of the items, different kinds of mistakes were classified. Very little free scope for subjective assessment was, however, allowed.

The principle of inter-judge reliability was not only applied to the items that required straight forward answers but also to those that required full solutions. Independently, two markers assessed the answers and solutions of the test items. The correspondence of these assessments is a measure of the inter-judge reliability. The measures for the inter-judge reliability were 98 % for grade four and 97 % for grade eight. These two measures are considered very high.

These assessments are the basis for the statistical analyses. Also, a number of pupils' solutions altogether has been analysed in order to create a picture of high precision of the nature of single pupils' mathematical knowledge. The analyses of the pupils' solutions in the TIMSS-project and in the national assessment test do not affect the reliability of the present study, since the data consist in the collected pupils' solutions. Instead, the analyses of the pupils' data are part of the validity assessment.

Construction validity describes to what degree the pupils' solutions were possible to categorize and whether the created categories covered all the pupils' solutions. Since only exposed understanding of the concepts and applications of the calculation procedures have been studied, incomplete solutions have been reported as non-categorized. This makes the content validity high.

External validity describes to what degree the result is possible to generalise to the population as a whole. By means of the released items only, the number of pupils of the two samples, grade four and eight in TIMSS, roughly corresponds to 40 % of the entire sample. This is not sufficiently large for an independent generalization to the whole population. If, however, the TIMSS-results are in harmony with relevant research and also with the results from the national assessment test, more valid conclusions will be possible to draw about the entire population of pupils. Clear tendencies in the results are also with higher certainty possible to extrapolate to the population.

A larger amount of pupils' solutions would of course have been both desirable and preferable. Subsequently the external validity would have increased. Within the given time limits, it would not, however, have been possible to analyse a still larger number of solutions.

11.5 Future Research

Along with the every-year performance of the national assessment test, there would be possibilities to change certain types of items in order to get a more detailed picture of the pupils' mathematical knowledge. As a suggestion, a significantly larger proportion of pupils' solutions would be collected. A sample of the size of 5000 would naturally make the conclusions about the population much more reliable and valid. Multiple-choice items would simplify the assessment and would as in the TIMSS-project reflect both known misconceptions and applications of the calculation procedures in incorrect contexts. Also single pupils' comprehensive solutions would be important to include in the analyses. Moreover, the pupils' solutions of the national assessment tests for grade 5 and 9 would be of interest to study. Not least the domain of algebra would be of special importance, since Swedish pupils' achievements in algebra need to be raised systematically.

Also TIMSS-data from other countries would be of a great magnitude in relation to the Swedish results. Does the character of the pupils' mistakes differ in main aspects or are the similarities striking? An interesting point of departure for the analyses would be to compare procedural and conceptual approaches in the teaching and thereby make a comparison between the present study and the would-be results of the nature of the pupils' mathematical knowledge.

Since the use of the conceptual models in the teaching is a decisive part of the results of the present study, the use of the models would be the centre of a more thorough study. In some test items, a conceptual model was applied outside its application range and led to unsuccessful answers. Therefore, one core issue would be to study how the different application ranges are taught and how they are described in the text books. At a simultaneous use of several conceptual models, it would in addition be of interest to study not only whether the pupils can keep them apart but also the effects of the manifoldness of models on their conceptual understanding.

Educational Measures

12. Educational Measures

In the following sections, a survey of the obstacles that have been found to interfere with the pupils' mathematical development will be given. These interfering obstacles argue for a more active role of the teacher in the classroom. Also the procedural approach of the teaching of geometry in grade four will be dealt with. After that the necessary requirements for the understanding of the formal algebra in grade eight will be emphasized. Finally, the advantage of a more conceptual teaching approach in geometry will be put forward.

12.1 More Instructing and Guiding Teachers

Pupils' understanding of the number concept and of the calculation procedures has not been sufficiently developed. Subtraction that requires trading showed to be especially problematic. All the three studies confirmed this position. Single pupils showed to have several parallel calculation procedures out of which some were incorrectly applied. But these pupils also exposed correctly applied strategies. In order to develop number facts, incorrect strategies have to be cleaned out. Having developed number facts results in an optimal use of working memory, namely to encode a text problem, to decide what calculation operations to use etc. and not to load it with the calculation of for instance 6 + 7. This can be illustrated by an example. If for example 5 + 3 now and then becomes 7, 8 or 9, there are no correct combinations to be stored in the long-term memory. It is necessary that 5 + 3 always is 8, so that this combination will be stored in the long-term memory and can be easily fetched. Therefore, the pupils must be offered opportunities to discuss mathematics in class with their teachers to get confirmation and feed back on their understanding. Independent work does not give this required and vital feed back. The project in Lilla Edet showed that the textbooks did not always describe the calculations strategies adequately or correctly. Therefore, teacher instruction and guidance is fundamental.

There is a risk that primary pupils in school practise and use incorrect strategies which become progressively established during their early school years. The pupils who mastered the standard algorithms for the four elementary operations were also those who were most successful in solving these kinds of items.

Encoding text problems into mathematical models seemed to be difficult for many pupils. There are reasons to believe that the different types of problem situations to be modelled by addition, subtraction, multiplication or division, have not been systematically taught. In view of the relatively low solving frequencies of this type of items, comparison situations that lead to subtraction have probably not been taught. Not many problem situations were represented in the test items, six in addition and subtraction and four in multiplication and division. The encoding of problem situations also included the concepts of partitive and quotitive division both of which showed to be very difficult for many pupils. When the encoding resulted in a quotitive division, the solving frequencies were low. When the encoding ended in a partitive division, however, the solving frequencies were higher.

More systematic teaching of these types of problem situations would probably improve the pupils' achievements dramatically.

Judging from the result of the present study fractions has not been taught in spite of the fact that they are part of the attainment targets for grade 5. The lack of teaching was confirmed by the low solving frequencies.

12.2 Abandon the Procedural Approach in the Teaching of Geometry!

As was found and is reported previously, a pupil can have several parallel ways of understanding one and the same concept. Among these, also misconceptions exist, which in certain situations misguide the pupil. Besides teacher instruction and guidance, discussions in class with supervision of the teacher can give the pupils chances to improve and develop their mathematical conceptions. This together with confirmation of correct mathematical conceptions, the pupils' misconceptions will gradually vanish for the benefit of the correct ones (Spitze, 1996). Such a conceptual teaching approach would have been advantageous also in the western countries, where the procedural approach is common. The low solving frequencies, especially on the items of contexts unaccustomed to the pupils, could be due to the procedural teaching approach. Since conceptual knowledge can generate procedural knowledge and since procedural knowledge only exceptionally can generate conceptual knowledge, there is much to gain from a more conceptual teaching approach (Rittle-Johnson & Wagner Alibali, 1999). Therefore, teachers need to learn the two different theories for conceptual learning, both of which have their roots in neuro-science. The two main theories are "theory revision" and "re-description" (Bentley, 2008a).

Conceptions of the area concept are crucial for understanding geometry. The area concept is often mixed up with the perimeter concept. Pupils need high-frequent experiences of the area conservation principle and of the additive character of the area concept. International research shows that procedural teaching does not result in mathematical understanding of the character of the area concept. From the results was confirmed, however, that the pupils seemed to have understood and mastered the principle of measurement.

The pupils' spatial ability was tested by means of two-dimensional representations of three-dimensional objects. It is known that this ability develops relatively late in schooling, which was also verified by the solving frequency. There are countries where computer animation is taken use of for developing the pupils' spatial ability. According to international research, there is evidence that specially designed educational computer software helps in developing spatial ability.

Despite the fact that rotation is a crucial part of the models of the angle concept that is used in comprehensive school, less than half of the pupils succeeded in solving the items on rotation. A reasonable question to ask is whether teachers are familiar with the three different models of the angle concept including their weaknesses. As have been pointed out in research report many times, teachers need to know what they teach.

12.3 Necessary Prerequisite for the Understanding of Formal Algebra

Eight-graders in general have a dynamic way of understanding the equality sign due to their elementary arithmetic training. For the understanding of algebra the static way is required. A dynamic understanding of the equality sign makes the solving procedures of equations not only complicated but also more or less impossible in contrast to a static understanding. The static way of understanding can easily be practised by number sentences like $5 + 3 = \square - 4$.

Mastered arithmetic knowledge constitutes a necessary basis for pupils' development of the understanding of formal algebra. Rather than taking arithmetic structures as a point of departure for simplifications of expressions, there are teachers, who use the Object Model. The application range of the Object Model is very limited, since it is only valid for additive simplifications. Having learnt this model, pupils transfer it to multiplicative simplifications and thereby fail in these kinds of items.

Generally, substitution of negative integers in expressions is difficult for eight-graders to understand. Given that b is equal to -1, a majority of the pupils thought that $-b$ is also equal to -1. This mistake can be suspected to have its roots in inadequately developed arithmetic thinking.

Without doubt, pupils' understanding of the variable concept is most problematic for the eight-graders. Several test items became extra difficult to solve because the meaning of the involved variables was not correctly understood. Misconceptions of the variable concept like non-symbolic representation, digit representation, additive representation and concrete object representation were exposed. Moreover, the pupils had difficulties to identify the two different ways of understanding the variable namely as a specific unknown number and as a generalized number. Most often, the variable was wrongly understood as a specific unknown number and not as a generalized number.

As has been reported in the research review (See section 2.2), pupils normally have a number of different ways of understanding a concept. These various ways of understanding a concept are often contextually linked. The concept of variable is in this respect no exception. Described in the present study are the different ways of understanding various concepts that the pupils explicitly exposed in their solutions of the test items. This does not necessarily imply that the pupils lack correct understanding of the mathematical concepts. The pupils seem, however, uncertain of when a certain way of understanding the variable should be applied.

Pupils' different ways of understanding the variable concept ought to be the top priority content in in-service training courses for secondary comprehensive school teachers. If this problem of the understanding of the variable concept is not adequately treated in comprehensive school, it will lead to serious consequences for the understanding of the concepts of function and graph in upper secondary school.

12.4 A Conceptual Approach in the Teaching of Geometry

Not only the four-graders but also the eight-graders generally mixed up the two concepts of perimeter and area, which is a phenomenon known from international research. Too many pupils seem not familiar with the additive character of the area concept and are therefore not capable of making calculations of the areas of composite figures. Unfamiliarity of the additive character of the area is probably due to a lack of conceptual experiences, which in turn is due to a predominating procedural teaching approach.

When the formula for calculation of the area of a triangle was possible to apply directly, the solving frequency was high, but when conceptual adaptations were necessary, the frequency decreased. Knowledge of the area concept of the circle does not only comprise the formula but also the understanding of how many area units are needed to cover its surface. In addition, pupils must learn to know how many times larger the area of a circle with the radius r is than a square with the side r. The strictly procedural teaching approach seems to create unnecessary problems for the pupils to understand the concept of area.

The conceptual models of the angle concept are also internationally known to be difficult to understand. All the three models have disadvantages. But the pupils apparently need a conceptual image for their understanding. However, the pupils seem to know that the sum of the interior angles of a triangle is 180°. But fewer pupils know that half a turn also corresponds to 180°. If the teaching mostly focuses on procedures, conceptual understanding of the angle concept will probably not be developed.

Angles are classified both to position and size. Vertical, alternate, interior and exterior are concepts that are linked to the position of the angle. Another character linked to position is that an angle can face a certain side. An isosceles triangle consists in one top angle and two base angles. The base angles are of equal size. The classification concerning the size of the angle is right, obtuse and acute. Because of the moderate solving frequency, the terminology linked to the size of an angle seems unknown to the pupils. Therefore, the angle terminology needs to be taught recurrently.

Spatial experiences in mathematics teaching seem to be decisive for the understanding and interpretation of two-dimensional representations of three-dimensional objects. A teaching approach that focuses computation does not prioritize these kinds of experiences. As has been pointed out in the account on geometry for grade four, computer animation is used in some countries for improving the development of pupils' spatial ability. The results seem promising.

Too much independent pupil work in class is a plausible reason for the results of the present study. Teaching without confirmations on what conceptions are right and what are wrong does not supply the pupils with possibilities to develop their mathematical conceptual knowledge. If a pupil has both correct and incorrect procedures and correct and incorrect ways of understanding the various mathematical concepts, the correct needs to be confirmed so that the incorrect gradually disappear. Not getting feed back, the incorrect procedures and the incorrect ways of understanding the concepts will be even more firmly established.

Conceptual teaching would surely be more advantageous and gainful for the pupils.

References

13. References

Adams, J., W. & Hitch, G., J. (1998). Children's Mental Arithmetic and Working Memory. In *The Development of Mathematics Skills*, Donlan, C., (Ed.), Studies in Developmental Psychology. London: Psychology Press.

Af Ekenstam, A. & Greger, K. (1983). Some Aspects of Children's Ability to Solve Mathematical Problems. *Educational Studies in Mathematics.* No. 14, pp. 369–384.

Arzt, A. & Armour-Thomas, E. (1999). A Cognitive Model for Examining Teachers' Instructional Practice in Mathematics: A Guide for Facilitating Teacher Reflexion. *Educational Studies in Mathematics.* Vol. 40, pp. 211–235.

Baddeley, A., D. (1986). *Working Memory.* Oxford, UK: Oxford University Press.

Baddeley, A., D. & Hitch, G., J. (1974). Working Memory. In *The Psychology of Learning and Motivation.* Bower, G. (Ed.), Vol. 8, pp. 47–90.

Behr, M., Erlwanger, S. & Nichols, E. (1980). How Children View the Equals Sign. *Mathematics Teaching*, No. 92, pp. 13–15.

Bell, A., Fischbein, E. & Greer, B. (1984). Choice of Operation in Verbal Arithmetic Problems: The Effect of Number Size, Problem Structure and Context. *Educational Studies in Mathematics.* No. 15, pp. 129–147.

Ben-Chaim, D., Lappan, G., & Houang, R., T. (1988). *American Educational Research Journal*, Vol. 25, No. 1, pp. 51–71.

Bentley, C., G. (2002). *The Roots of Variation in English Teaching.* Göteborg, Studies in Educational Sciences, 176. Göteborg: Acta Universitatis Gothoburgensis.

Bentley, P-O. (2008a). *Mathematics Teachers and Their Conceptual Models. A New Field of Research.* Göteborg, Studies in Educational Sciences, 265. Göteborg: Acta Universitatis Gothoburgensis.

Bentley, P-O. (2008b). Pupils' Arithmetic Knowledge and the Procedural Models in their Teaching. (In Press).

Booth, L, R. (1984). *Algebra: Children's Strategies and Errors.* New Winsor, Berkshire, England: NFER-Nelson Publishing Co.

Carpenter, T. P., Franke, M. L. & Levi, L. (2003). *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School.* Portsmouth, NH: Heinemann.

Chick, H., L. & Baker, M., K. (2005). In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education.* Vol. 2, pp. 249–256. Melbourne: PME.

Clements, D., H. & Stephan, M. (2003). Measurement in PreK-2 Mathematics. In *Mathematics: Standards for Early Childhood Mathematics.* Clements, D,. H., Sarama, J. & DiBiase, A-M. (Eds.) Lawrence Erlbaum Assoc Inc.

Close, G., S. (1982). *Children's Understanding of Angle at the Primary/Secondary Transfer Stage*. Polytechnic of the South Bank, London.

Cuneo, D. (1980). A General Strategy for Quantity Judgements: The Height + Width Rule. *Child Development*. No. 51, pp. 299–301.

Davis, B. (1997). Listening for Differences: An Evolving Conception of Mathematics Teaching. *Journal of Research in Mathematics Education*. Vol. 28, No. 3, pp. 355–376.

DeStefano, D., & LeFevre, J-A. (2004). The Role of Working Memory in Mental Arithmetic. *European Journal of Cognitive Psychology*. No. 16(3), pp. 353–386.

Douady, R. & Perrin, M-J. (1986). Concerning Conceptions of Area (pupils aged 9 to 11). *Proceedings of PME 10*, London, 253–258.

Falkner, K. P., Levi, L. & Carpenter, T. P. (1999). Children's Understanding of Equality: A Foundation for Algebra. *Teaching Children Mathematics*, No. 6(4), pp. 232–236.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The Role of Implicit Models in Solving Verbal Problems in Multiplication and Division. *Journal for Research in Mathematics Education*, No. 16, pp. 3–17.

Foxman, D., & Ruddock, G. (1984). Concepts and Skills: Line Symmetry and Angle. *Mathematics in Schools*, March 1984, 9–13.

Foxman, D., & Beishuizen, M., (2002). Mental Calculation Methods Used by 11-Year-Olds in Different Attainment Bands: A Reanalysis of Data from the 1987 APU Survey in the UK. In *Educational Studies in Mathematics*, 51 pp. 41–69.

Freudenthal, (1973). *Mathematics as an educational task*. Riedel, Dordrecht.

Fuson, K., C. (1992). Addition and Subtraction. In Grouws, D., A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillian Publishing Company.

Fuson, K., C., Wearne, D., Hiebert, J., C., Murray, H., G., Human, P., G., Carpenter, T., P., & Fennema, E. (1997). Children's Conceptual Structure for Multi-Digit Numbers and Methods of Multi-Digit Addition and Subtraction. In *Journal for Research in Mathematics Education*, 28, pp. 130–162.

Gelman, R. & Gallistel, C., R. (1978). *The Child's Understanding of Number*. London: Harvard University Press.

Ginsburg, H. (1977). *Children's Arithmetic*. New York: Van Nostrand.

Graeber, A., O., Tirosh, D. & Glover, R. (1989). Preservice Teachers' Misconceptions in Solving Verbal Problems in Multiplication and Division. *Journal for Research in Mathematics Education*. No. 20, pp. 324–345.

Greer, B. (1992). Multiplication and Division as Models of Situations. In Grouws, D., A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillian Publishing Company.

Hart, K., M. (1981). Ratio and Proportion. In Hart, K., M. (Ed.) *Children's Understanding of Mathematics*: 11–16, London: John Murray.

Hiebert, J. (1981). Units of Measure: Results and Implications from National Assessment. *Arithmetic Teacher*, 28 (6), 38–43.

Johansson, B., S. (2005). Numeral Writing Skill and Elementary Arithmetic Mental Calculations. *Scandinavian Journal of Educational Research.* Vol. 49, No. 1, pp. 3–25.

Kaput, J. (1985). *Multiplicative Word Problem and Intensive Quantities: An Integrated Software Response.* (Technical Report 85-19). Cambridge, MA: Harvard University, Educational Technology Centre.

Kieran, C. (1981). Concepts Associated with the Equality Symbol. *Educational Studies in Mathematics*, No. 12, pp. 317–326.

Klein, A., S. & Beishuizen, M. (1998). The Empty Number Line in Dutch Second Grades: Realistic Versus Gradual Program Design. *Journal for Research in Mathematics Education*, 29, pp. 443–464.

Kordaki, M. & Potari, D. (1997). *Children's Approaches to Area Measurement through Different Contexts.* Manuscript submitted for publication.

Krainer, K. (1989). *'Lebendige Geometrie: Überlegungen zu einem integriven Verständnis von Geometrieunterricht anhand des Winkelbegriffs'* [Living Geometry: Deliberations on a Comprehensive Understanding of Geometry Teaching as Exemplified by the Angle Concept], Lang, Frankfurt.

Küchemann, D., E. (1981). Algebra. In Hart, K., M. (Ed.) *Children's Understanding of Mathematics*: 11–16, London: John Murray.

Lindahl, M. (1996). *Inlärning och erfarande. Ettåringars möte med förskolans värld.* Göteborg: Acta Universitatis Gothoburgensis.

Lo, J.-J., Gaddis, K., and Henderson, D. (1996) 'Building upon Student Experience in a College Geometry Course', *For the Learning of Mathematics*, 16(1), 34–40.

Logie, R., H. (1995). *Visual-Spatial Working Memory.* Hove, UK: Lawrence Erlbaum Associates Ltd.

MacGregor, M. & Stacey, K. (1997). Students' Understanding of Algebraic Notation. *Educational Studies in Mathematics.* No. 33. pp. 1–19.

Mangan, C. (1986). *Choice of Operation Multiplication and Division Word Problems.* Unpublished Doctorial Dissertation, Queen's University, Belfast.

Marton, F. & Booth, S. (2000). *Om lärande.* Lund: Studentlitteratur.

Mitchelmore, M. C. (1989) 'The Development of Children's Concepts of Angle', in G. Vergnaud (ed.), *Proceedings of the 13th International Conference on the Psychology of Mathematics Education*, Paris, Vol. 2, pp. 304–311.

Mitchelmore, M. C., & White, P. (1998). Development of Angle Concepts: A Framework for Research, *Mathematics Education Research Journal*, 10(3), 4–27.

Mitchelmore, M. C. & White, P. (2000) Development of Angle Concepts by Progressive Abstraction and Generalisation. *Educational Studies in Mathematics*, 41(3); pp. 209–238

Patronis, T. & Thomaidis, Y. (2008). On Arithmetization of School Geometry in the Setting of Modern Axiomatic, *Science and Education.*

Philipp, R. (1992). The Many Uses of Algebraic Variables. In *The Mathematics Teacher*. Vol. 85, No. 7. pp. 557–561.

Piaget J. , Inhelder B. & Sheminska A. (1981). *The Child's Conception of Geometry*, N.Y: Norton & Company.

Rittle-Johnson, B. & Wagner Alibali, M. (1999). Conceptual and Procedural Knowledge of Mathematics: Does One Lead to the Other? *Journal of Educational Psychology*. Vol. 91. No. 1. pp. 175–189.

Roels, G. (1985). 'Het fenomeen hoek' [The Angle Phenomenon], *Wiskunde en Onderwijs*, 11, 127–138.

Ryu, H., A., Chong, Y., O. & Song, S., H. (2007). In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 137–144. Seoul: PME.

Schweiger, F. (1986) 'Winkelbegriff und Winkelmaß' [The Angle Concept and Angle Measurement], *Mathematik im Unterricht*, 11, 1–9.

Seyler, D., J., Kirk, E., P., & Ashcraft, M., H. (2002). *Elementary Subtraction*. Manuscript under review.

Skolöverstyrelsen (1966). *Matematikterminologi i skolan*. Skolöverstyrelsen: SÖ-förlaget.

Spitzer, M., (1996). *Geist im Netz: Modelle für Lernen, Denken und Handeln*. Spektrum Akademischer Verlag, Heidelberg

Stigler, J., W. & Hiebert, J. (1999). *The Teaching Gap*. New York: The Free Press.

Strehl, R. (1983). 'Anschauliche Vorstellung und mathematische Theorie beim Winkelbegriff' [Visualisation and Mathematical Theory of the Angle Concept], *Mathematica Didactica*, 6, 129–146.

Swedner, H. (1978). *Sociologisk metod. En bok om kunskapsproduktion och förändringsarbete*. Lund: Liber Läromedel.

Thompson, I. (1999). Mental Calculation Strategies for Addition and Subtraction, Part 1. *Mathematics in School*, November, pp. 2–4.

Usiskin, Z. (1998). Paper-and-Pencil Algorithms in a Calculator and Computer Age. In Morrow, l., J. & Kenny, M., J. (Eds.), *The Teaching and Learning of Algorithms in School Mathematics* (Yearbook of the national Council of Teachers of Mathematics, pp. 7–20) Reston, VA: NCTM.

Van den Heuvel-Panhuizen, (2001). Realistic Mathematics Education in the Netherlands, in *Principles and Practices in Arithmetic Teaching*, Anghileri, J (Ed), Buckingham: Open University Press, pp. 15–31.

Wagner, S. (1983). What Are Those Things Called Variables? *Mathematics Teacher*, Vol. 76, pp. 474–479.

Vygotsky, L. (1986). *Thought and Language*. London: The MIT Press.

Yackel, E. (2001). Perspectives on Arithmetic from Classroom-Based Research in the United States of America, in *Principles and Practices in Arithmetic Teaching*, Anghileri, J. (Ed.), Open University Press, Buckingham, pp. 15–32.

TIMSS (Trends in International Mathematics and Science Study) is a study of students' knowledge in mathematics and science in grades 4 and 8.

In this report students' answers to TIMSS-items in mathematics are analyzed. The focus is on the content domains where Swedish students do less well. These domains are Number and Geometric shapes and Measures in grade 4, Algebra and Geometry in grade 8. The analysis aim to show how well students understand key mathematical concepts and can apply calculation-procedures in these domains.

The analysis is carried out and the report is written by Per-Olof Bentley, associate professor and PhD at the University of Gothenburg, as responsible for mathematics education in the Swedish TIMSS 2007-project. The author is responsible for the contents and the statements expressed.

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